

A Free Boundary Value Problem For The Stationary Navier-Stokes Equations

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A Free Boundary Value Problem For The Stationary Navier-Stokes Equations

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Prof. Klaus Schmitt

Prof. Heinz-Otto Peitgen

*Oh snail climb mount Fuji
but slowly, slowly...*

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Abstract

An incompressible viscous fluid flow over a flat plate is considered. It is assumed that the flow is stationary, periodic in the two horizontal directions and has a free upper surface. Knowing a basic solution, the flow is considered as a bifurcation problem with the Reynolds number as bifurcation parameter. The whole problem is split up into a problem on a fixed domain and the normal stress equation. Existence and uniqueness results are derived for both problems separately. After this a fixed point formulation can be given for the whole problem. Numerical computations for the case of a two dimensional flow have been carried out.

Introduction

We want to consider an incompressible, viscous fluid flowing over a flat plate which is in the x, y - plane. We assume that when observing the fluid flow from a frame of references, which is moving with some speed c , the fluid is time independent and periodic in x and y . The speed c is a wave velocity in the x, y - plane, i.e. $c = (c_1, c_2, 0)$. Furthermore the upper surface Γ is a free boundary. The flow is driven by the outer pressure gradient $-\partial p_a / \partial x = a$.

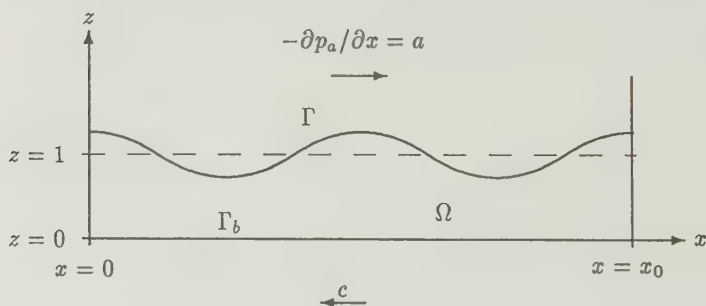


Figure 0.1: *Fluid flow over a flat plate.*

At the free surface the tangential and the normal stress conditions need to be satisfied, where tangential and normal are to be understood with respect to the free surface. A nonzero stress in tangential direction would cause a free surface to change. Hence the tangential stress has to be zero. The stress in normal direction has a jump discontinuity when crossing the free surface. The jump is proportional to the mean curvature of the surface and the proportionality factor is the surface tension of the fluid. A third boundary condition at the free surface is that the velocity of a stationary flow has to be in the tangent plane.

Free boundary fluid flow problems have been studied constantly throughout the last 100 years. A large portion of this work has been concerned with two dimensional problems. In the case of two dimensions the methods differ considerably from those for three dimensions. In two dimensions methods from complex analysis can be used. For the Stokes problem, e.g., the stream function is a harmonic function, which has to satisfy Bernoulli's equation. Two dimensional problems of this type have been considered e.g. by Zeidler [Zei], where one can find a detailed reference list.

For the three dimensional problem one has to apply methods which are particular for Partial Differential Equations and Functional Analysis. For Navier-Stokes problems with a fixed domain those methods can be found in the books by Ladyzhenskaya [Lad] and Temam [Tem], e.g.. One of the most important contributions for this kind of free boundary problems is the one by Solonnikov and Scadilov [Sol/Sca]. In [Sol/Sca] Solonnikov and Scadilov consider the free boundary Stokes problem but without the normal stress condition. Instead of using the normal stress condition they assume that the domain Ω is known. If we write

$$\partial\Omega = \Gamma_1 + \Gamma_2,$$

where Γ_2 represents the free boundary part of $\partial\Omega$, the problem Solonnikov and Scadilov consider is the following:

$$\left. \begin{aligned} -\nu\Delta u + \text{grad } p &= f \\ \text{div } u &= 0 \\ u|_{\Gamma_1} &= 0 \\ u \cdot n|_{\Gamma_2} &= 0 \\ \tau \cdot t(u) \cdot n|_{\Gamma_2} &= 0. \end{aligned} \right\} \quad \text{in } \Omega$$

Here we have used the following notation:

- u - velocity,
- p - pressure,
- n - outer unit normal on Γ ,
- τ - any tangential vector on Γ ,
- ν - kinematic viscosity and
- $t(u)$ - stress tensor with $t_{ik} = -p\delta_{ik} + \nu(\partial_k u_i + \partial_i u_k)$.

Solonnikov and Scadilov prove that this problem has a unique solution. This leads to the fact that the full free boundary problem can be split up into

the problem above and the normal stress equation. This approach can be used for our problem as well. Another important reference in this area is the work of Bemelmans [Bem1] and [Bem2]. Bemelmans considers the fluid motion inside a drop and is concerned with the question if the problem can be solved uniquely for perturbations of the forcing term. Bemelmans gives a positive answer to this question [Bem2] using Zehnder's version of the hard implicit function theorem [Zeh]. In connection with the drop problem the classical work of Lichtenstein [Lich] ought to be mentioned also. Lichtenstein is concerned with equilibria figures of fluid bodies.

Here we consider the fluid flow as a bifurcation problem, where we take the Reynolds number as bifurcation parameter.

In Chapter 1 we compute the basic solution (u_0, p_0, Γ_0) and derive the equations for the perturbed problem. Under the perturbed problem we understand the problem where the trivial solution corresponds to the basic solution (u_0, p_0, Γ_0) . In order to do this we have to transform the problem first onto a specific domain, since the domain Ω depends on the solution. In Section 1.2 we carry out the transformation of the basic solution onto the domain Ω on which the solution for the perturbed problem is defined.

In Chapter 2 we assume that the domain Ω is known and we consider the functional analysis of the problem without the normal stress condition. In Section 2.1 we follow the treatment of Tartar [Tar] by showing that the pressure $p \in L^2(\Omega)$ is uniquely determined up to a constant by the velocity $u \in J \subset (H^1(\Omega))^3$ (see Def. 2.2). This is shown in [Tar] for Ω being a C^2 -domain. We prove that this result holds also if Ω is only a $C^{0,1}$ -domain. In Section 2.2 we show that the bilinear form

$$E(u, \varphi) = \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial \varphi_i}{\partial x_k} + \frac{\partial \varphi_k}{\partial x_i} \right) dx$$

defines an inner product on J where the induced norm is equivalent to the H^1 -norm. In Section 2.3 we show the equivalence between the weak solution and the solution in the distributional sense for this problem. A solution u is called a weak solution, if u satisfies a particular integral equation (see Def. 2.30).

In Chapter 3 we still assume that the domain Ω is known and we still leave the normal stress equation unconsidered. In Section 3.1 we derive from the weak formulation that a unique solution u exists for the Stokes and the linearized problem. We consider both problems with homogeneous

Dirichlet boundary data on the bottom Γ_b of Ω . Since for our problem the boundary datum ϕ_0 is not zero, we extend ϕ_0 to all of Ω in such a way that the extension is divergence free. This extension is constructed by using the method given by Ladyzhenskaya [Lad]. We show that we do not need necessarily a C^2 - domain Ω but that in our case a $C^{0,1}$ - domain is sufficient for the construction. Then we may write the solution u of the nonhomogeneous Stokes problem as

$$u = u' + \phi_0,$$

where u' is a solution of a homogeneous Stokes problem and ϕ_0 the extension of the boundary data. For the existence proof of the linearized problem we use already the regularity result that a solution $u \in (H^2(\Omega))^3$, which we show in Section 3.2. Proofs for the regularity of the Stokes problem can be found in the literature (see [Sol/Sca]). We present here a proof without any representation formulas and concentrate on those parts which differ from a standard regularity proof one might find in a textbook (see e.g. [Wlo]).

In Chapter 4 now we consider the full problem included the normal stress equation and give a fixed point formulation for it. The normal stress equation is an equation of the prescribed mean curvature type, which has been studied in Differential Geometry (see e.g. [Trei]) in greater depth. The existence of a solution for a prescribed mean curvature equation can be shown only for right hand sides which are bounded by some constant. We give estimates for the solution u and the right hand side of the normal stress equation $n \cdot t(u) \cdot n|_{\Gamma}$ such that a solution (u, ζ) is equivalent to a fixed point of a certain operator. Here the function $\zeta(x, y)$ determines the free boundary Γ . Moreover we show that the linearization of this operator is a Fredholm operator of index zero.

In Chapter 5 we consider the two dimensional problem and transfer the equations into a system of ordinary differential equations (ODE) by using a Fourier expansion for the stream function. Then we apply the program AUTO [Doe/Ker] to this ODE- system. AUTO is written for solving bifurcation problems for ODEs and algebraic equations. We compute the problem for the case where we take one Fourier coefficient besides the constant term. For this case we find a bifurcation point and a set of nontrivial solutions branching off from the bifurcation point. In the figures at the end of this chapter we trace out the velocities and the free surface for some nontrivial solutions.

Chapter 1

The basic solution and the perturbed problem

In this chapter we compute first a basic solution to our problem which one might find in many fluid mechanics textbooks, e.g. in [Yih]. The velocity has parabolic profile with zero at the bottom and its maximum at the free boundary. The free boundary is the plane $\{z = 1\}$. The maximum of the basic velocity depends linearly on the outside pressure gradient $\partial_x p$. If the pressure gradient increases it might happen that from some value on waves are starting to occur on the surface and the velocity profile will change. We shall describe the deviation of the velocity from the basic velocity. Since this velocity is defined on a different domain we need to transform all the equations into one domain. We shall derive transformations for writing the problem on the domain Ω_0 , the domain where the basic solution u_0 is defined, as well as on the new domain Ω . The final goal in this chapter is to obtain the equations for the perturbation velocity.

The full problem is described by the following set of equations:

$$\left. \begin{aligned} -\nu \Delta u + \text{grad } p + u_k \partial_k u &= f_G \\ \text{div } u &= 0 \end{aligned} \right\} \quad \text{in } \Omega \quad (1.1)$$

$$\left. \begin{aligned} \partial^\alpha u|_{x=0} &= \partial^\alpha u|_{x=x_0} \\ \partial^\alpha u|_{y=0} &= \partial^\alpha u|_{y=1} \end{aligned} \right\} \quad |\alpha| \leq 1 \quad (1.2)$$

$$u|_{\Gamma_b} = -c \quad (1.3)$$

$$u \cdot n|_{\Gamma} = 0 \quad (1.4)$$

$$\tau \cdot t(u) \cdot n|_{\Gamma} = 0 \quad (1.5)$$

$$n \cdot t(u) \cdot n|_{\Gamma} = -p_a + 2\kappa H, \quad (1.6)$$

where

u - velocity	f_G - gravitational force
p - pressure	p_a - pressure outside the fluid
n - outer unit normal on Γ	τ - any tangential vector on Γ
ν - kinematic viscosity	κ - surface tension/density
$2H$ - mean curvature of the free boundary Γ	
$t(u)$ - stress tensor: $t_{ik} = -p\delta_{ik} + \nu(\partial_k u_i + \partial_i u_k)$	
$u_k \partial_k u = u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$	

The boundary condition (1.3) is the non-slip condition for viscous fluids in the moving frame of references. Boundary condition (1.5) is due to the fact that in a stationary flow there can not be any tangential stresses. The normal stress on a free surface (1.6) is equal to the normal stress outside the fluid plus a term which is proportional to the mean curvature. For a more detailed discussion of these equations see e.g. [Weh/Lai].

1.1 The basic solution (P_0, U_0)

In the case the free boundary is a flat plane parallel to the x, y plane a solution of this problem is known. Let us assume that Γ is a part of the plane $\{z = 1\}$. The fluid then fills out the box Ω_0 . Let us assume furthermore, that

- (i) U_0 is only a function of z ,
- (ii) $\text{grad } P_0 = (-a, 0, -g)$ with $a \equiv \text{const.}$ and g is the gravitational acceleration,
- (iii) $p_a = -ax$.

Then the solution can be computed in a straightforward way.

- (a) Because of (i) we can write $U_0 = U_0(z) = (U_1(z), U_2(z), U_3(z))$.

Then $\text{div } U_0 = dU_3/dz = 0$ implies $U_3 \equiv \text{const.}$

From the boundary condition (1.3) one can conclude that

$$U_0 \cdot n|_{z=1} = U_3 = 0.$$

- (b) Integrating (ii) the pressure becomes $P_0 = -ax - gz + b_1$, where b_1 is a constant.

- (c) Now the Navier Stokes equations can be reduced to:

$$-\nu(d^2 U_1/dz^2) - a = 0 \quad \text{and} \quad -\nu(d^2 U_2/dz^2) = 0.$$

By integrating these equations we get

$$U_1(z) = (a/2\nu)z^2 + d_1 z + d_2 \quad \text{and} \quad U_2(z) = e_1 z + e_2$$

with d_i and e_i ($i=1,2$) being constants.

(d) The boundary condition (1.3) now implies

$$U_1(0) = d_2 = -c_1 \quad \text{and} \quad U_2(0) = e_2 = -c_2.$$

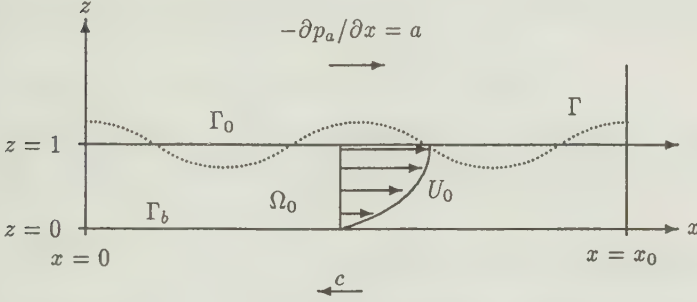


Figure 1.1: The basic solution U_0 .

(e) For computing the constants d_1 and e_1 we use the tangential and normal stress conditions (1.5) and (1.6). The functional matrix of U_0 is

$$DU_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{a}{\nu}z + d_1 & e_1 & 0 \end{pmatrix}$$

and so the stress tensor becomes

$$\begin{aligned} t(U_0) &= -P_0I + \nu(DU_0 + (DU_0)^t) \\ &= \begin{pmatrix} ax + gz - b_1 & 0 & -az + \nu d_1 \\ 0 & ax + gz - b_1 & \nu e_1 \\ -az + \nu d_1 & \nu e_1 & ax + gz - b_1 \end{pmatrix} \end{aligned}$$

Hence the stress vector $t(U_0) \cdot n$ is

$$t(U_0) \cdot n = \begin{pmatrix} -az + \nu d_1 \\ \nu e_1 \\ ax + gz - b_1 \end{pmatrix} \stackrel{(z=1)}{=} \begin{pmatrix} -a + \nu d_1 \\ \nu e_1 \\ ax + g - b_1 \end{pmatrix}.$$

So the tangential stresses on the free boundary are

$$\tau_1 \cdot t \cdot n = -a + \nu d_1 = 0 \quad \text{and} \quad \tau_2 \cdot t \cdot n = \nu e_1 = 0.$$

And thus $d_1 = a/\nu$ and $e_1 = 0$.

The normal stress condition becomes $n \cdot t \cdot n = ax + g - b_1 = ax$. Since the surface Γ is given as $\{z = 1\}$, its mean curvature H is of course 0. This gives us $b_1 = g$.

Hence we have computed the basic solution to be:

$$U_0 = \begin{pmatrix} \frac{a}{2\nu} z(2-z) - c_1 \\ -c_2 \\ 0 \end{pmatrix} \quad (1.7)$$

$$P_0 = -ax + g(1-z) \quad (1.8)$$

1.2 The perturbed problem

After finding the solution (U_0, P_0) for the domain Ω_0 we want to know what will happen, if this flow is perturbed. Are there solutions of the form $(u + U_0, p + P_0)$ on a domain Ω , where Ω might be different from Ω_0 ? Since we can not assume that $\Omega = \Omega_0$, it is not clear at all what is meant by $u + U_0$ and $p + P_0$. This will be defined in the following way: Let us assume that there exist transformations

$$\Phi : \Omega_0 \longrightarrow \Omega$$

and

$$\Psi = \Phi^{-1} : \Omega \longrightarrow \Omega_0.$$

Then U_0 and P_0 can be transformed into functions u_0 and p_0 on Ω . Hence we can write the problem in the following way:

$$\left. \begin{aligned} -\nu \Delta u + \text{grad } p + u_{0k} \partial_k u + u_k \partial_k u_0 + u_k \partial_k u &= f \\ \text{div } u &= 0 \end{aligned} \right\} \quad \text{in } \Omega \quad (1.9)$$

$$\left. \begin{aligned} \partial^\alpha u + u_0|_{x=0} &= \partial^\alpha u + u_0|_{x=x_0} \\ \partial^\alpha u + u_0|_{y=0} &= \partial^\alpha u + u_0|_{y=1} \\ u + u_0|_{\Gamma_b} &= -c \\ (u + u_0) \cdot n|_\Gamma &= 0 \\ \tau_k \cdot t(u + u_0) \cdot n|_\Gamma &= 0, \quad k = 1, 2 \\ n \cdot t(u + u_0) \cdot n|_\Gamma &= -p_a + 2\kappa H. \end{aligned} \right\} \quad |\alpha| \leq 1 \quad (1.10)$$

Furthermore we want to require an additional boundary condition for p :

$$\begin{aligned} p|_{x=0} &= p|_{x=x_0} \\ p|_{y=0} &= p|_{y=1} \end{aligned} \quad (1.11)$$

In order to write down the boundary problem for u we first have to study the transformations Φ and Ψ .

1.2.1 The transformation $\Phi : \Omega_0 \longrightarrow \Omega$

We want to restrict ourselves to small perturbations of Ω_0 . Therefore we may assume that Ω_0 can be mapped onto Ω in the following way:

$$\Phi : \Omega_0 \longrightarrow \Omega,$$

defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z(1 + \zeta(x, y)) \end{pmatrix}. \quad (1.12)$$

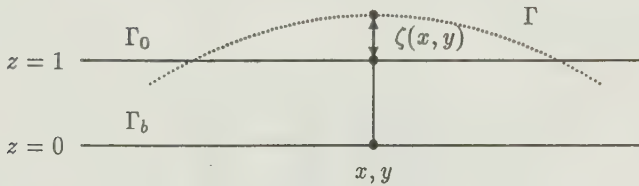


Figure 1.2: The transformation map $\Phi : \Omega_0 \rightarrow \Omega$.

The function ζ is defined in the obvious way (see Fig.3), i.e.:

$\Gamma = \text{graph}(1 + \zeta)$. Knowing Φ we are going to compute now the functional matrix, the metric tensor, the Christoffel symbols and the normal and tangential unit vectors on Γ . The notation used here for derivatives is $\zeta_x = \partial_x \zeta$.

Functional matrix

$$D\Phi = \begin{pmatrix} 1 & 0 & z\zeta_x \\ 0 & 1 & z\zeta_y \\ 0 & 0 & 1 + \zeta \end{pmatrix} = \frac{\partial \Phi^i}{\partial x^j}. \quad (1.13)$$

Metric tensor

The elements of the metric tensor are given by $g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^3$.

$$g_{ij} = \begin{pmatrix} 1 + (z\zeta_x)^2 & z^2\zeta_x\zeta_y & z\zeta_x(1+\zeta) \\ z^2\zeta_x\zeta_y & 1 + (z\zeta_y)^2 & z\zeta_y(1+\zeta) \\ z\zeta_x(1+\zeta) & z\zeta_y(1+\zeta) & (1+\zeta)^2 \end{pmatrix} \quad (1.14)$$

and the inverse matrix g^{ij} is given by

$$g^{ij} = \begin{pmatrix} 1 & 0 & \frac{-z\zeta_x}{1+\zeta} \\ 0 & 1 & \frac{-z\zeta_y}{1+\zeta} \\ \frac{-z\zeta_x}{1+\zeta} & \frac{-z\zeta_y}{1+\zeta} & \frac{1 + (z\zeta_x)^2 + (z\zeta_y)^2}{(1+\zeta)^2} \end{pmatrix}. \quad (1.15)$$

Furthermore

$$g \stackrel{def}{=} \det(g_{ij}) = (1+\zeta)^2 \quad (1.16)$$

Christoffel symbols

$$\Gamma_{jk}^1 = \Gamma_{jk}^2 = 0 \quad (1.17)$$

$$\Gamma_{jk}^3 = \begin{pmatrix} \frac{z\zeta_{xx}}{1+\zeta} & \frac{z\zeta_{xy}}{1+\zeta} & \frac{\zeta_x}{1+\zeta} \\ \frac{z\zeta_{xy}}{1+\zeta} & \frac{z\zeta_{yy}}{1+\zeta} & \frac{\zeta_y}{1+\zeta} \\ \frac{\zeta_x}{1+\zeta} & \frac{\zeta_y}{1+\zeta} & 0 \end{pmatrix}. \quad (1.18)$$

1.2.2 The transformation $\Psi : \Omega \longrightarrow \Omega_0$

$$\Psi : \Omega \longrightarrow \Omega_0$$

is defined by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \hline 1 + \zeta(\bar{x}, \bar{y}) \end{pmatrix} \quad (1.19)$$

The metric tensor and the Christoffel symbols which are given by Ψ are called h_{ij} and Δ_{jk}^i respectively.

Functional matrix

$$D\Psi = \begin{pmatrix} 1 & 0 & -\frac{\bar{z}\zeta_{\bar{x}}}{(1+\zeta)^2} \\ 0 & 1 & -\frac{\bar{z}\zeta_{\bar{y}}}{(1+\zeta)^2} \\ 0 & 0 & \frac{1}{1+\zeta} \end{pmatrix} = \frac{\partial\Psi^i}{\partial x^j}. \quad (1.20)$$

Metric tensor

$$h_{ij} = \begin{pmatrix} 1 + \frac{\bar{z}^2\zeta_{\bar{x}}^2}{(1+\zeta)^4} & \frac{\bar{z}^2\zeta_{\bar{x}}\zeta_{\bar{y}}}{(1+\zeta)^4} & -\frac{\bar{z}^2\zeta_{\bar{x}}}{(1+\zeta)^3} \\ \frac{\bar{z}^2\zeta_{\bar{x}}\zeta_{\bar{y}}}{(1+\zeta)^4} & 1 + \frac{\bar{z}^2\zeta_{\bar{y}}^2}{(1+\zeta)^4} & -\frac{\bar{z}^2\zeta_{\bar{y}}}{(1+\zeta)^3} \\ -\frac{\bar{z}^2\zeta_{\bar{x}}}{(1+\zeta)^3} & -\frac{\bar{z}^2\zeta_{\bar{y}}}{(1+\zeta)^3} & \frac{1}{(1+\zeta)^2} \end{pmatrix} \quad (1.21)$$

$$h^{ij} = \begin{pmatrix} 1 & 0 & \frac{\bar{z}\zeta_{\bar{x}}}{1+\zeta} \\ 0 & 1 & \frac{\bar{z}\zeta_{\bar{y}}}{1+\zeta} \\ \frac{\bar{z}\zeta_{\bar{x}}}{1+\zeta} & \frac{\bar{z}\zeta_{\bar{y}}}{1+\zeta} & \frac{(1+\zeta)^4 + \bar{z}^2(\zeta_{\bar{x}}^2 + \zeta_{\bar{y}}^2)}{(1+\zeta)^2} \end{pmatrix} \quad (1.22)$$

and

$$h \stackrel{def}{=} \det(h_{ij}) = \frac{1}{(1+\zeta)^2}. \quad (1.23)$$

Christoffel symbols

$$\Delta_{jk}^1 = \Delta_{jk}^2 = 0 \quad (1.24)$$

$$\Delta_{jk}^3 = \begin{pmatrix} \frac{(2\zeta_{\bar{x}}^2 - (1+\zeta)\zeta_{\bar{x}\bar{x}})\bar{z}}{(1+\zeta)^2} & \frac{(2\zeta_{\bar{x}}\zeta_{\bar{y}} - (1+\zeta)\zeta_{\bar{x}\bar{y}})\bar{z}}{(1+\zeta)^2} & -\frac{\zeta_{\bar{x}}}{1+\zeta} \\ \frac{(2\zeta_{\bar{x}}\zeta_{\bar{y}} - (1+\zeta)\zeta_{\bar{x}\bar{y}})\bar{z}}{(1+\zeta)^2} & \frac{(2\zeta_{\bar{y}}^2 - (1+\zeta)\zeta_{\bar{y}\bar{y}})\bar{z}}{(1+\zeta)^2} & -\frac{\zeta_{\bar{y}}}{1+\zeta} \\ -\frac{\zeta_{\bar{x}}}{1+\zeta} & -\frac{\zeta_{\bar{y}}}{1+\zeta} & 0 \end{pmatrix}. \quad (1.25)$$

Furthermore let us mention some facts which will be needed later on:

$$\left. \begin{aligned} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} &= \Gamma_{sk}^s = \Gamma_{3k}^3 \\ \frac{1}{\sqrt{h}} \frac{\partial \sqrt{h}}{\partial \bar{x}^k} &= \Delta_{sk}^s = \Delta_{3k}^3. \end{aligned} \right\} \quad (1.26)$$

1.2.3 The normal and tangential vectors to Γ

The free boundary Γ is the zero set of the function

$$F(x, y, z) = z - (1 + \zeta(x, y)).$$

The outer unit normal $\bar{n}$ is then given by

$$\bar{n} = \frac{\nabla F}{\sqrt{\|\nabla F\|}} \quad \text{or} \quad \bar{n}_i = \frac{F_{,i}}{\sqrt{\|\nabla F\|}}.$$

Computing ∇F leads to

$$\bar{n}_i = \begin{pmatrix} \frac{-\zeta_x}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}} \\ \frac{-\zeta_y}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}} \\ 1 \\ \sqrt{1 + \zeta_x^2 + \zeta_y^2} \end{pmatrix}. \quad (1.27)$$

Notice that $\bar{n}$ is a covariant vector. We transform $\bar{n}_i$ onto Γ_0 and call the transformed vector n_i . Before we do the transformation, let us recall how to transform covariant and contravariant vectors. If

$$\Omega_0 \xrightarrow{\Phi} \Omega, \quad x^i \mapsto y^i$$

is the one to one transformation mapping, then a covariant vector has to be transformed like

$$U_i \text{ on } \Omega_0 \mapsto u_i = \frac{\partial \Psi^j}{\partial y^i} U_j \text{ on } \Omega \quad (1.28)$$

and a contravariant vector like

$$U^i \text{ on } \Omega_0 \mapsto u^i = \frac{\partial \Phi^i}{\partial x^j} U^j \text{ on } \Omega. \quad (1.29)$$

Here we have used the summation convention, i.e.: one has to take the sum over those indices which appear both as an upper and a lower index. We shall use this convention throughout this chapter. So the transformed unit

normal vector is $n_i = \frac{\partial \Phi^j}{\partial x^i} \Big|_{z=1} \bar{n}_j$ or

$$n_i = \begin{pmatrix} 0 \\ 0 \\ \frac{1+\zeta}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \end{pmatrix}. \quad (1.30)$$

The contravariant unit normal can then be computed as $n^i = g^{ij} n_j$, which is the vector

$$n^i = \begin{pmatrix} \frac{-\zeta_x}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \\ \frac{-\zeta_y}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \\ \frac{\sqrt{1+\zeta_x^2+\zeta_y^2}}{1+\zeta} \end{pmatrix}. \quad (1.31)$$

For the computations in the next section we need to know also the unit tangential vectors $\bar{\tau}_k^i$ ($k=1,2$) at Γ and the transformed ones at Γ_0 . The unit tangential vectors are given by

$$\bar{\tau}_1^i = \frac{1}{\sqrt{1+\zeta_x^2}} \begin{pmatrix} 1 \\ 0 \\ \zeta_x \end{pmatrix} \quad (1.32)$$

and

$$\bar{\tau}_2^i = \frac{1}{\sqrt{1+\zeta_x^2} \sqrt{1+\zeta_x^2+\zeta_y^2}} \begin{pmatrix} -\zeta_x \zeta_y \\ 1+\zeta_x^2 \\ \zeta_y \end{pmatrix}. \quad (1.33)$$

For the transformations of the tangential vectors we apply the formula (1.29) and compute that

$$\tau_1^i = \frac{1}{\sqrt{1 + \zeta_x^2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (1.34)$$

$$(1.35)$$

and

$$\tau_2^i = \frac{1}{\sqrt{1 + \zeta_x^2} \sqrt{1 + \zeta_x^2 + \zeta_y^2}} \begin{pmatrix} -\zeta_x \zeta_y \\ 1 + \zeta_x^2 \\ 0 \end{pmatrix}. \quad (1.36)$$

The respective covariant vectors are then given by $(\tau_k)_i = g_{il}(\tau_k)^l$ which is

$$(\tau_1)_i = \frac{1}{\sqrt{1 + \zeta_x^2}} \begin{pmatrix} 1 + \zeta_x^2 \\ \zeta_x \zeta_y \\ (1 + \zeta) \zeta_x \end{pmatrix} \quad (1.37)$$

$$(1.38)$$

and

$$(\tau_2)_i = \frac{1}{\sqrt{1 + \zeta_x^2} \sqrt{1 + \zeta_x^2 + \zeta_y^2}} \begin{pmatrix} 0 \\ 1 + \zeta_x^2 + \zeta_y^2 \\ (1 + \zeta) \zeta_y \end{pmatrix}. \quad (1.39)$$

1.2.4 Transformation of the basic solution onto Ω

Transformation of P_0 :

$$p_0 \stackrel{def}{=} P_0(\Psi(\bar{x})) = -a\bar{x} + g(1 - \frac{\bar{z}}{1 + \zeta}). \quad (1.40)$$

The contravariant vectors U_0^i and f_G^i are going to be transformed by using (1.29):

$$\tilde{f}_G^i = \frac{\partial \Phi^i}{\partial \bar{x}^j} \cdot f_G^j(\Psi(\bar{x})) = (D\Phi)^t(\Psi(\bar{x})) \cdot \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

where

$$(D\Phi)^t(\Psi(\bar{x})) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\bar{z}\zeta_{\bar{x}}}{1+\zeta} & \frac{\bar{z}\zeta_{\bar{y}}}{1+\zeta} & 1+\zeta \end{pmatrix}.$$

Hence

$$\tilde{f}_G^i = \begin{pmatrix} 0 \\ 0 \\ -g(1+\zeta) \end{pmatrix}. \quad (1.41)$$

The transformation for U_0 leads to

$$\tilde{U}_0^i = \begin{pmatrix} \frac{a}{2\nu} \frac{\bar{z}}{1+\zeta} (2 - \frac{\bar{z}}{1+\zeta}) - c_1 \\ -c_2 \\ \frac{a}{2\nu} \frac{\bar{z}^2 \zeta_{\bar{x}}}{(1+\zeta)^2} (2 - \frac{\bar{z}}{1+\zeta}) - \frac{\bar{z}}{1+\zeta} (c_1 \zeta_{\bar{x}} + c_2 \zeta_{\bar{y}}) \end{pmatrix}. \quad (1.42)$$

But unfortunately $\text{div } \tilde{U}_0$ is not zero in the coordinates on Ω . Therefore we transform U_0 not into $\tilde{U}_0$ but into $u_0 = \sqrt{h} \tilde{U}_0$. That u_0 is a solenoidal vector field in the coordinates of Ω can be seen by taking the covariant derivatives. In terms of the covariant derivatives the equation $\text{div } U_0 = 0$ is preserved under the transformation (1.29), i.e.:

$$\tilde{U}_{0,i}^i = 0,$$

i.e.,

$$\tilde{U}_{0,i}^i = \frac{\partial \tilde{U}_0^i}{\partial \bar{x}^i} + \Delta_{is}^i \tilde{U}_0^s = \frac{\partial \tilde{U}_0^i}{\partial \bar{x}^i} + \frac{1}{\sqrt{h}} \frac{\partial \sqrt{h}}{\partial \bar{x}^s} \tilde{U}_0^s = 0.$$

In the last equation we have used (1.26). Now it is easy to see that u_0 as defined above is divergence free in Ω . And u_0 is given by

$$u_0^i = \begin{pmatrix} \frac{a}{2\nu} \left(\frac{2\bar{z}}{(1+\zeta)^2} - \frac{\bar{z}^2}{(1+\zeta)^3} \right) - \frac{c_1}{1+\zeta} \\ -\frac{c_2}{1+\zeta} \\ \frac{a}{2\nu} \left(\frac{2\bar{z}^2 \zeta_{\bar{x}}}{(1+\zeta)^3} - \frac{\bar{z}^3 \zeta_{\bar{x}}}{(1+\zeta)^4} \right) - \frac{c_1 \bar{z} \zeta_{\bar{x}} + c_2 \bar{z} \zeta_{\bar{y}}}{(1+\zeta)^2} \end{pmatrix}. \quad (1.43)$$

1.2.5 Boundary conditions for u

The periodic boundary conditions (1.2) are satisfied by u_0 , since by the very choice of ζ they are satisfied by Ψ and $D\Psi$ as well. Therefore we can write for u :

$$\left. \begin{aligned} \partial^\alpha u|_{x=0} &= \partial^\alpha u|_{x=x_0} \\ \partial^\alpha u|_{y=0} &= \partial^\alpha u|_{y=1} \end{aligned} \right\} \quad |\alpha| \leq 1. \quad (1.44)$$

From (1.3) it can be computed that $u_0|_{\Gamma_b} = -c/(1+\zeta)$. Hence the boundary condition $u|_{\Gamma_b}$ becomes:

$$u|_{\Gamma_b} = -c \frac{\zeta}{1+\zeta} \stackrel{def}{=} \phi_0. \quad (1.45)$$

For computing $u_0 \cdot n|_{\Gamma}$ we write it in tensor notation, which is invariant under our transformations:

$$u_0^i \bar{n}_i|_{\Gamma} = \frac{1}{\sqrt{g}} U_0^i n_i|_{\Gamma_0} = 0.$$

This means that orthogonal vectors are transformed into orthogonal vectors. So we obtain

$$u \cdot n|_{\Gamma} = 0. \quad (1.46)$$

Now we have to compute the boundary conditions for the normal and the tangential components of the stress vector. Let us begin by writing the tangential stress condition (1.5) in tensor notation:

$$\left(-(p + p_0)g^{ik} + 2\nu D^{ik}(u + u_0) \right) \bar{\tau}_k \bar{n}_i = 0.$$

where g^{ij} is the metric tensor (1.15) and

$$2D^{ik}(v) = g^{kl}v^i_{,l} + g^{il}v^k_{,l} = \frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i} \quad \text{in } \Omega.$$

The transformation of 1.5 to Γ_0 becomes:

$$\left(-(p + P_0)g^{ik} + 2\nu D^{ik}(u + \frac{1}{\sqrt{g}}U_0) \right) \tau_k n_i = 0, \quad (1.47)$$

where now

$$2D^{ik}(v) = \left(g^{kl} \frac{\partial v^i}{\partial y^l} + g^{il} \frac{\partial v^k}{\partial y^l} \right) + \left(g^{kl} \Gamma_{ls}^i v^s + g^{il} \Gamma_{ls}^k v^s \right) \quad \text{in } \Omega_0.$$

Here we have called the coordinates in Ω_0 y_1, y_2 and y_3 . The tangential stress condition above can now be rewritten in the form:

$$\left(-pg^{ik} + 2\nu D^{ik}(u)\right) \tau_k n_i + \left(-P_0 g^{ik} + 2\nu D^{ik}\left(\frac{1}{\sqrt{g}}U_0\right)\right) \tau_k n_i = 0. \quad (1.48)$$

Claim 1.1

Let $\tau(1) = (1, 0, 0)$ and $\tau(2) = (0, 1, 0)$. $\tau(1)$ and $\tau(2)$ span the tangent plane to Γ_0 . Suppose that τ is any tangent vector to Γ_0 , then there are $\alpha, \beta \in \mathbb{R}$ such that $\tau = \alpha\tau(1) + \beta\tau(2)$ and

$$\left(-P_0 g^{ik} + 2\nu D^{ik}\left(\frac{1}{\sqrt{g}}U_0\right)\right) \tau_k n_i = -(\alpha\gamma_1 + \beta\gamma_2),$$

where

$$\gamma_1 = -2 \frac{(\frac{a}{2} - c_1\nu)\zeta_{xx} - c_2\nu\zeta_{xy}}{\sqrt{1 + \zeta}\sqrt{1 + \zeta_x^2 + \zeta_y^2}}$$

and

$$\gamma_2 = -2 \frac{(\frac{a}{2} - c_1\nu)((1 + \zeta_x^2)\zeta_{xy} - \zeta_x\zeta_y\zeta_{xy}) - c_2\nu((1 + \zeta_x^2)\zeta_{yy} - \zeta_x\zeta_y\zeta_{yy})}{(1 + \zeta)(1 + \zeta_x^2 + \zeta_y^2)\sqrt{1 + \zeta_x^2}}.$$

Proof:

First we observe that

$$-P_0 g^{ik} n_i \tau_k = -P_0 n^k \tau_k = 0.$$

Let us now compute $2D^{ik}(\frac{1}{\sqrt{g}}U_0)$:

$$\begin{aligned} 2D^{ik}\left(\frac{1}{\sqrt{g}}U_0\right) &= g^{kl} \frac{\partial(\frac{1}{\sqrt{g}}U_0^i)}{\partial y^l} + g^{il} \frac{\partial(\frac{1}{\sqrt{g}}U_0^k)}{\partial y^l} + g^{kl}\Gamma_{ls}^i \frac{1}{\sqrt{g}}U_0^s + g^{il}\Gamma_{ls}^k \frac{1}{\sqrt{g}}U_0^s \\ &= \frac{1}{\sqrt{g}}(g^{kl} \frac{\partial U_0^i}{\partial y^l} + g^{il} \frac{\partial U_0^k}{\partial y^l}) + g^{kl} \frac{\partial(\frac{1}{\sqrt{g}})}{\partial y^l} U_0^i + g^{il} \frac{\partial(\frac{1}{\sqrt{g}})}{\partial y^l} U_0^k + \\ &\quad \frac{1}{\sqrt{g}}(g^{kl}\Gamma_{ls}^i U_0^s + g^{il}\Gamma_{ls}^k U_0^s). \end{aligned}$$

That U_0 satisfies the boundary condition (1.5) at Γ_0 means:

$$-P_0 \delta_k^i n_i \tau^k + \nu \left(\frac{\partial U_0^i}{\partial y^k} n_i \tau^k + \frac{\partial U_0^k}{\partial y^i} n_i \tau_k \right) = 0$$

or

$$\underbrace{-P_0 g^{ik} n_i \tau_k}_{=0} + \nu \left(\underbrace{g^{lk} g_{ks} \frac{\partial U_0^i}{\partial y^l} n_i \tau^s}_{g^{lk} \frac{\partial U_0^i}{\partial y^l} n_i \tau_k} + \underbrace{g^{li} g_{is} \frac{\partial U_0^k}{\partial y^l} n^s \tau_k}_{g^{li} \frac{\partial U_0^k}{\partial y^l} n_i \tau_s} \right).$$

So we can write:

$$\left. \begin{aligned} & \left(-P_0 g^{ik} + 2\nu D^{ik} \left(\frac{1}{\sqrt{g}} U_0 \right) \right) n_i \tau_k = \\ & -\frac{\nu}{g} \left(g^{kl} \frac{\partial \sqrt{g}}{\partial y^l} U_0^i + g^{il} \frac{\partial \sqrt{g}}{\partial y^l} U_0^k \right) n_i \tau_k \\ & + \frac{\nu}{\sqrt{g}} \left(g^{kl} \Gamma_{ls}^i U_0^s + g^{il} \Gamma_{ls}^k U_0^s \right) n_i \tau_k \end{aligned} \right\} \quad (1.49)$$

We analyze now each term on the right hand side. The first term in (1.49) can be rewritten by using (1.26), (1.17) and (1.18) as

$$\frac{\nu}{g} g^{kl} \frac{\partial \sqrt{g}}{\partial y^l} U_0^i n_i \tau_k = \frac{\nu}{\sqrt{g}} \Gamma_{3l}^3 U_0^i n_i \tau^l.$$

It can be seen that this term is zero using the following argument:

$$\Gamma_{3l}^3 U_0^i = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the dots represent elements which are possibly nonzero. The special form of the matrix has been obtained because $U_0^3 = 0$ and $\Gamma_{33}^3 = 0$. This leads to $\Gamma_{3l}^3 U_0^i n_i \tau^l = 0$, since

$$(0, 0, \bullet) \cdot \begin{pmatrix} \blacksquare & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix} = 0.$$

The second term in the first sum can be computed in the same way and we obtain

$$\frac{\nu}{g} g^{il} \frac{\partial \sqrt{g}}{\partial y^l} U_0^k n_i \tau_k = \frac{\nu}{\sqrt{g}} \underbrace{\Gamma_{3l}^3 U_0^k g^{lr} g_{jk} n_r \tau^j}_M,$$

where the matrix M is of the form

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ 0 & 0 & 0 \end{pmatrix}.$$

This implies that this term is also zero. The only nonzero term left in (1.49) is now the last term, which can be rewritten as

$$\frac{\nu}{\sqrt{g}} \left(g^{kl} \Gamma_{ls}^i U_0^s + g^{il} \Gamma_{ls}^k U_0^s \right) n_i \tau_k = 2 \frac{\nu}{\sqrt{g}} \Gamma_{ls}^3 U_0^s n_3 \tau^l.$$

Taking now $\tau = \tau(i)$ ($i=1,2$) leads to

$$\frac{\nu}{\sqrt{g}} \left(g^{kl} \Gamma_{ls}^i U_0^s + g^{il} \Gamma_{ls}^k U_0^s \right) n_i \tau(i)_k = \gamma_i,$$

where the γ_i have been defined in the claim. This proves the claim. ■

We observe that the functions γ_i ($i=1,2$) are independent of z . So we can write down the boundary conditions for the tangential stress components using Claim 1.1 and (1.48):

$$\tau(k) \cdot t(u) \cdot n|_{\Gamma} = \begin{cases} \gamma_1 & \text{for } k = 1 \\ \gamma_2 & \text{for } k = 2 \end{cases}. \quad (1.50)$$

Now we turn to the last boundary condition, the normal stress condition (1.6):

$$n \cdot t(u + u_0) \cdot n|_{\Gamma} = n \cdot t(u) \cdot n|_{\Gamma} + n \cdot t(u_0) \cdot n|_{\Gamma} = -p_a + 2\kappa H.$$

We do the computations in the same way as we did it for the tangential components. We transform the equation onto Γ_0 and use the results we

have established in the proof of Claim 1.1:

$$\begin{aligned}
 t^{ik}(u_0)n_in_k &= \left(-P_0g^{ik} + 2\nu D^{ik}\left(\frac{1}{\sqrt{g}}U_0\right) \right) n_in_k \\
 &= -P_0g^{ik}n_in_k + \frac{\nu}{\sqrt{g}}(g^{kl}\frac{\partial U_0^i}{\partial y^l} + g^{il}\frac{\partial U_0^k}{\partial y^l})n_in_k \\
 &\quad - \frac{\nu}{g}\left(g^{kl}\frac{\partial\sqrt{g}}{\partial y^l}U_0^i + g^{il}\frac{\partial\sqrt{g}}{\partial y^l}U_0^k\right)n_in_k \\
 &\quad + \frac{\nu}{\sqrt{g}}\left(g^{kl}\Gamma_{ls}^iU_0^s + g^{il}\Gamma_{ls}^kU_0^s\right)n_in_k.
 \end{aligned} \tag{1.51}$$

From the the computations following (1.49) we can conclude that the third term on the right hand side of (1.51) is zero. The first and second terms of (1.51) can be computed easily:

$$\begin{aligned}
 -P_0g^{ik}n_in_k &= -P_0|_{\Gamma_0}n^kn_k = -p_a \\
 \left(g^{kl}\frac{\partial U_0^i}{\partial y^l} + g^{il}\frac{\partial U_0^k}{\partial y^l}\right)n_in_k &= 0.
 \end{aligned}$$

The only nonzero term of (1.51) is therefore the last term which is

$$\frac{\nu}{\sqrt{g}}\left(g^{kl}\Gamma_{ls}^iU_0^s + g^{il}\Gamma_{ls}^kU_0^s\right)n_in_k = 2\frac{\nu}{\sqrt{g}}\Gamma_{ls}^3U_0^sn_3n^l.$$

This implies

$$\begin{aligned}
 n \cdot t(u_0) \cdot n|_{\Gamma} &= 2\frac{(\frac{a}{2} - c_1\nu)(\zeta_x + \zeta_y)\zeta_{xx} - c_2\nu(\zeta_x + \zeta_y)\zeta_{yy}}{(1 + \zeta)(1 + \zeta_x^2 + \zeta_y^2)} \\
 &\quad - 2\frac{(\frac{a}{2} - c_1\nu)\zeta_x - c_2\nu\zeta_y}{(1 + \zeta)^2} \\
 &\stackrel{def}{=} -\beta(x, y),
 \end{aligned}$$

or

$$n \cdot t(u) \cdot n|_{\Gamma} = \beta + 2\kappa H. \tag{1.52}$$

Let us summarize the boundary conditions for u by collecting the computed functions in a definition and writing down the equations for the free boundary problem in terms of u and the new defined functions.

Definition 1.2

$$\begin{aligned}\gamma_1 &= -2 \frac{(\frac{a}{2} - c_1\nu)\zeta_{xx} - c_2\nu\zeta_{xy}}{\sqrt{1+\zeta}\sqrt{1+\zeta_x^2+\zeta_y^2}} \\ \gamma_2 &= -2 \frac{(\frac{a}{2} - c_1\nu)((1+\zeta_x^2)\zeta_{xy} - \zeta_x\zeta_y\zeta_{xy}) - c_2\nu((1+\zeta_x^2)\zeta_{yy} - \zeta_x\zeta_y\zeta_{xy})}{(1+\zeta)(1+\zeta_x^2+\zeta_y^2)\sqrt{1+\zeta_x^2}} \\ \beta &= \beta_0 + \beta_1,\end{aligned}$$

where

$$\begin{aligned}\beta_0 &= -2 \frac{(\frac{a}{2} - c_1\nu)(\zeta_x + \zeta_y)\zeta_{xx} - c_2\nu(\zeta_x + \zeta_y)\zeta_{yy}}{(1+\zeta)(1+\zeta_x^2+\zeta_y^2)} \\ \beta_1 &= 2 \frac{(\frac{a}{2} - c_1\nu)\zeta_x - c_2\nu\zeta_y}{(1+\zeta)^2} \\ \phi_0 &= -c \frac{\zeta}{1+\zeta}.\end{aligned}$$

For later purposes we want to define β_{ik} for $(i=0,1)$ and $(k=1,2)$:

$$\begin{aligned}\beta_0 &= \beta_{01} + \beta_{02} & \beta_1 &= \beta_{11} + \beta_{12} \\ \beta_{01} &= -a \frac{(\zeta_x + \zeta_y)\zeta_{xx}}{(1+\zeta)(1+\zeta_x^2+\zeta_y^2)} \\ \beta_{02} &= 2 \frac{c_1\nu(\zeta_x + \zeta_y)\zeta_{xx} + c_2\nu(\zeta_x + \zeta_y)\zeta_{yy}}{(1+\zeta)(1+\zeta_x^2+\zeta_y^2)} \\ \beta_{11} &= a \frac{\zeta_x}{(1+\zeta)^2} \\ \beta_{12} &= -2 \frac{c_1\nu\zeta_x - c_2\nu\zeta_y}{(1+\zeta)^2}.\end{aligned}$$

■

Now we can write down the equations for the perturbations (u, p, ζ) :

$$\left. \begin{aligned} -\nu\Delta u + \text{grad } p + u_{0k}\partial_k u + u_k\partial_k u_0 + \dot{u}_k\partial_k u &= f \\ \text{div } u &= 0 \end{aligned} \right\} \quad \text{in } \Omega, \quad (1.53)$$

and the boundary conditions

$$\left. \begin{aligned}
 & \left. \begin{aligned}
 \partial^\alpha u|_{x=0} &= \partial^\alpha u|_{x=x_0} \\
 \partial^\alpha u|_{y=0} &= \partial^\alpha u|_{y=1}
 \end{aligned} \right\} |\alpha| \leq 1 \quad (B1) \\
 & p|_{x=0} = p|_{x=x_0} \\
 & p|_{y=0} = p|_{y=1} \\
 & u|_{\Gamma_b} = \phi_0 \quad (B2) \\
 & u \cdot n|_{\Gamma} = 0 \quad (B3) \\
 & \tau_k \cdot t(u) \cdot n|_{\Gamma} = \gamma_k, \quad k = 1, 2 \\
 & n \cdot t(u) \cdot n|_{\Gamma} = \beta + 2\kappa H,
 \end{aligned} \right\} \quad (1.54)$$

where

$$f = f_G - (-\nu \Delta u_0 + \text{grad } p_0 + u_{0k} \partial_k u_0). \quad (1.55)$$

Chapter 2

The functional analysis for the perturbed problem

In this chapter we define and discuss the function spaces which we shall use in the next chapters. First we define a subspace V of $H^1(\Omega)$ in which we include the boundary conditions (B1), (B2) and (B3) of (1.54).

Definition 2.1

Let us define $\dot{V} \subset (C^\infty(\Omega))^3$ by:

$$\dot{V} \stackrel{def}{=} \{v \in (C^\infty(\Omega))^3 : \partial^\alpha v|_{x=0} = \partial^\alpha v|_{x=x_0} ; \partial^\alpha v|_{y=0} = \partial^\alpha v|_{y=1} , \\ \forall |\alpha| \leq 1; \ v|_{\Gamma_b} = 0; \ v \cdot n|_{\Gamma} = 0\}$$

and V the closure of $\dot{V}$ in the norm of $H^1(\Omega)$, i.e.,

$$V \stackrel{def}{=} \overline{\dot{V}}^{\|\cdot\|_{H^1}}.$$

The norm $\|\cdot\|_{H^1}$ is the one which is induced by the inner product

$$\langle v, u \rangle_{H^1} = \sum_{k=1}^3 \int_{\Omega} v \cdot u \, dx + \sum_{i,k=1}^3 \int_{\Omega} \frac{\partial v^i}{\partial x^k} \frac{\partial u^i}{\partial x^k} \, dx = \langle v, u \rangle_{L^2} + \langle \nabla v, \nabla u \rangle_{L^2}.$$

For the space V we shall define a new inner product, which will turn out to be equivalent to $\|\cdot\|_{H^1}$. For showing the existence we can follow the

approach which has been taken by Ladyzhenskaya [Lad] for the Dirichlet problem and by Solonnikov/Scadilov [Sol/Sca] for the Stokes problem with a free boundary. The problem can be transformed into a weak formulation of the form

$$\frac{\nu}{2} E(u', \varphi) = l(\varphi), \quad \forall \varphi \in J, \quad (2.1)$$

where $J \subset V$ is the subspace of divergence free vector fields, $u = u' + \phi_0$ with $u' \in J$ and ϕ_0 an extension of the Dirichlet boundary data ϕ_0 from Γ_b to Ω .

Definition 2.2

$$J \stackrel{def}{=} \{v \in V : \operatorname{div} v = 0\}.$$

We shall show that:

$u \in V$ is a solution of (1.53) in the distributional sense is equivalent to $u = u' + \phi_0$ and $u' \in J$ satisfies (2.1).

2.1 Function spaces

In this section we are going to summarize and develop some results about Sobolev spaces. We will follow the treatment of Tartar [Tar]. Since the domain Ω is only of class $C^{0,1}$ (see Def. 2.8) and not C^2 we have to generalize the results of [Tar] to this case. If not otherwise stated the domain Ω will always be a bounded domain in $\mathbf{R}^3$.

Definition 2.3

Let $(\mathcal{D}(\Omega))^3$ be the set of all vector valued C^∞ -functions with compact support in Ω . Define

$$\mathcal{V} \stackrel{def}{=} \{v \in (\mathcal{D}(\Omega))^3 : \operatorname{div} v = 0\}.$$

Let us briefly recall the definition of the Sobolev spaces $H^{-m}(\Omega)$, $m \in \mathbf{N}$ (see e.g. [Tar, p.16] or [Adm, p.47 ff.]).

Definition 2.4

Let $m \in \mathbb{N}$, then $H^{-m}(\Omega)$ is defined as the dual space of $H_0^m(\Omega)$ in the L^2 -duality $\langle \cdot, \cdot \rangle \stackrel{\text{def}}{=} \langle \cdot, \cdot \rangle_{L^2}$. In this duality we identify the dual of L^2 with itself. $\varphi \in \mathcal{D}'(\Omega)$ is an element of $H^{-m}(\Omega)$ iff

$$\varphi(u) \stackrel{\text{def}}{=} \langle \varphi, u \rangle \stackrel{\text{def}}{=} \int_{\Omega} \varphi u \, dx, \quad \forall u \in H_0^m(\Omega),$$

defines a linear functional on $H_0^m(\Omega)$.

Remark 2.5

$\varphi \in H^{-m}(\Omega)$ can be characterized in the following way:

$\forall \alpha \in \mathbb{N}^3$ with $|\alpha| \leq m$, $\exists \varphi_{\alpha} \in L^2(\Omega)$ such that $\varphi = \sum_{|\alpha| \leq m} \varphi_{\alpha}$ and

$$\langle \varphi, u \rangle_{L^2} = \sum_{|\alpha| \leq m} \int_{\Omega} \varphi_{\alpha} D^{\alpha} u \, dx, \quad \forall u \in H_0^m(\Omega).$$

The following space will be of main interest for us in this chapter.

Definition 2.6

$$X(\Omega) \stackrel{\text{def}}{=} \{u \in H^{-1}(\Omega) : \frac{\partial u}{\partial x_i} \in H^{-1}(\Omega), (i = 1, 2, 3)\}.$$

By Remark 2.5 it is obvious that $L^2(\Omega) \subset X(\Omega)$. We can show that under some conditions the inverse inclusion is also true, i.e.: $X(\Omega) = L^2(\Omega)$. The following result is given in [Tar, p.26 ff.].

Lemma 2.7

$X(\Omega) = L^2(\Omega)$ in the following three cases:

- (i) $\Omega = \mathbb{R}^3$
- (ii) $\Omega = \mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_3 > 0\}$
- (iii) Ω is a C^2 -domain (see Def.2.8 below).

We remark that (iii) implies already the other two cases. But in order to show (iii) one proves first (i) and (ii).

Definition 2.8

A bounded domain $\Omega \subset \mathbb{R}^3$ is called a C^k -domain or a domain of class C^k , $k \in \mathbb{N}$, iff

$\exists$ open sets $U_i \subset \mathbb{R}^3$ and mappings $\eta_i : U_i \rightarrow \mathbb{R}^3$, $\forall i \in I$, I an index set, such that

(i) $\bigcup_{i \in I} U_i \supset \Omega$

(ii) $\eta_i : U_i \rightarrow \mathbb{R}^3$ are C^k -diffeomorphisms

(iii) $\eta_i(U_i \cap \partial\Omega) \subset \{x \in \mathbb{R}^3 : x_3 = 0\} = \partial\mathbb{R}_+^3$, $\forall i$ with $U_i \cap \partial\Omega \neq \emptyset$.

The pair $(U_i, \eta_i)_{i \in I}$ is called a C^k -covering.

We shall show that (iii) in Lemma 2.7 remains true if the domain Ω is only Lipschitz, i.e. the mappings η_i in Def. 2.8 are Lipschitz continuous. Before showing this we give the proof of (iii) in Lemma 2.7 for Ω being a C^2 -domain.

Proof of Lemma 2.7 (iii):

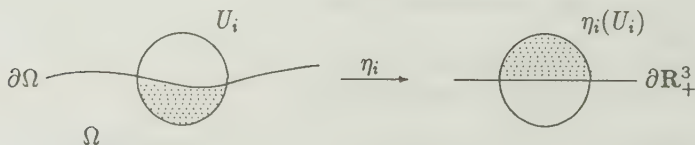
Let $(U_i, \eta_i)_{i \in I}$ be a C^2 -covering of Ω and $(\theta_i)_{i \in I}$ be a partition of the unity subordinate to the covering above, i.e., the support of θ_i , $\text{supp}\theta_i \subset U_i$.

Case 1: $U_i \subset \Omega$

If $u \in X(\Omega)$, then $\theta_i u \in X(\Omega)$ and $\theta_i u$ can be extended to $\mathbb{R}^3$ by setting it equal to zero outside Ω . Hence

$$\theta_i u \in X(\mathbb{R}^3) = L^2(\mathbb{R}^3) \subset L^2(\Omega).$$

Case 2: $U_i \cap \partial\Omega \neq \emptyset$



If $u \in X(\Omega)$, then $\theta_i u \in X(\Omega)$. Now we shall apply the following lemma:

Lemma 2.9

Let Ω_1, Ω_2 be two bounded domains in $\mathbf{R}^3$, $\eta : \Omega_1 \longrightarrow \Omega_2$ a C^2 - diffeomorphism with $\eta \in C^2(\overline{\Omega}_1, \overline{\Omega}_2)$ and $\eta^{-1} \in C^2(\overline{\Omega}_2, \overline{\Omega}_1)$, then:

- (i) $u \in H^{-1}(\Omega_2) \mapsto u \circ \eta \in H^{-1}(\Omega_1)$,
- (ii) $u \in L^2(\Omega_2) \mapsto u \circ \eta \in L^2(\Omega_1)$.

Define

$$V_i \stackrel{def}{=} \eta_i(U_i \cap \Omega), \quad \partial V_i = \partial V_{i,1} + \partial V_{i,2} \quad \partial V_{i,1} = \partial V_i \cap \partial \mathbf{R}_+^3$$

Applying the lemma above we can conclude that

$$\theta_i u \circ \eta^{-1} \in X(V_i)$$

and moreover that

- (i) $\text{supp}(\theta_i u \circ \eta^{-1}) \cap \partial V_{i,1} \neq \emptyset$ (in the general case)
- (ii) $\text{supp}(\theta_i u \circ \eta^{-1}) \cap \partial V_{i,2} = \emptyset$.

Because of (ii) $\theta_i u \circ \eta^{-1}$ can be extended to the halfspace $\mathbf{R}_+^3$ by setting it equal to zero outside V_i . This implies that

$$\theta_i u \circ \eta^{-1} \in X(\mathbf{R}_+^3) = L^2(\mathbf{R}_+^3) \subset L^2(V_i).$$

If we apply the lemma above once more, we obtain that

$$\theta_i u = (\theta_i u \circ \eta_i^{-1}) \circ \eta_i \in L^2(U_i \cap \Omega).$$

Since $\theta_i u$ has compact support in U_i ,

$$\theta_i u \in L^2(\Omega) \quad \implies u = \sum_{i \in I} \theta_i u \in L^2(\Omega),$$

since the sum is locally finite. ■

It remains to proof Lemma 2.9.

Proof of Lemma 2.9:

For $u \in H^{-1}(\Omega_2)$ we define

$$\langle u \circ \eta, f \rangle \stackrel{def}{=} \langle u, (f \circ \eta^{-1}) \det |D\eta^{-1}| \rangle, \quad f \in H_0^1(\Omega_1). \quad (2.2)$$

The definition of $\langle u \circ \eta, f \rangle$ is the expected one as one can see by taking $u \in L^2(\Omega_2)$. Then $u \circ \eta$ means the usual composition of functions. By the transformation rule for integrals we can write for $u \in L^2(\Omega_2)$

$$\langle u \circ \eta, f \rangle = \int_{\Omega_1} (u \circ \eta) f \, dx = \int_{\Omega_2} u(f \circ \eta^{-1}) |\det D\eta^{-1}| \, dy.$$

If

$$(f \circ \eta^{-1}) |\det D\eta^{-1}| \in H_0^1(\Omega_2),$$

then the second integral is equal to

$$\langle u, (f \circ \eta^{-1}) |\det D\eta^{-1}| \rangle.$$

Claim 2.10

If $f \in H_0^1(\Omega_1)$, then $f \circ \eta^{-1} \in H_0^1(\Omega_2)$. Moreover

$$\eta^{-1} : H^1(\Omega_1) \longrightarrow H^1(\Omega_2)$$

defines a bounded linear operator by the mapping $f \mapsto f \circ \eta^{-1}$.

Proof of Claim 2.10:

By the transformation rule for integrals we can compute

$$\int_{\Omega_2} (f \circ \eta^{-1})^2 \, dy = \int_{\Omega_1} f^2 |\det D\eta| \, dx.$$

Since $f \in L^2(\Omega_1)$ and $|\det D\eta| \in C^0(\overline{\Omega}_1)$,

$$\int_{\Omega_1} f^2 |\det D\eta| \, dx \leq \max_{\overline{\Omega}_1} |\det D\eta| \int_{\Omega_1} f^2 \, dx < \infty.$$

We remark that we actually needed only that η and η^{-1} are $\in C^1(\overline{\Omega}_1)$ and $\in C^1(\overline{\Omega}_2)$, respectively. Hence we have obtained:

$$\|f \circ \eta^{-1}\|_{L^2(\Omega_2)}^2 \leq \text{const.} \|f\|_{L^2(\Omega_1)}^2. \quad (2.3)$$

We shall show now that the derivatives $\partial_{y_i}(f \circ \eta^{-1})$ belong also to $L^2(\Omega_2)$, where y_i ($i = 1, 2, 3$) are the coordinates in Ω_2 .

$$\begin{aligned}
\int_{\Omega_2} \left(\frac{\partial}{\partial y_i} (f \circ \eta^{-1}) \right)^2 dy &= \int_{\Omega_2} \left(\sum_{k=1}^3 \frac{\partial f}{\partial x_k} (\eta^{-1}(y)) \frac{\partial \eta_k^{-1}}{\partial y_i} (y) \right)^2 dy \\
&\leq \text{const.} \sum_{k=1}^3 \int_{\Omega_2} \left(\frac{\partial f}{\partial x_k} (\eta^{-1}(y)) \right)^2 \left(\frac{\partial \eta_k^{-1}}{\partial y_i} (y) \right)^2 dy \\
&\leq \text{const.} \sum_{k=1}^3 \int_{\Omega_1} \left(\frac{\partial f}{\partial x_k} \right)^2 \left(\frac{\partial \eta_k^{-1}}{\partial y_i} (\eta(x)) \right)^2 \left| \det \frac{\partial \eta}{\partial x} \right| dx \\
&\leq \text{const.} \sum_{k=1}^3 \int_{\Omega_1} \left(\frac{\partial f}{\partial x_k} \right)^2 dx.
\end{aligned}$$

Here we have used that all the partial derivatives of η and η^{-1} are elements of $C^0(\bar{\Omega}_1)$ and $C^0(\bar{\Omega}_2)$, respectively. So we have shown:

$$\|f \circ \eta^{-1}\|_{H^1(\Omega_2)}^2 \leq \text{const.} \|f\|_{H^1(\Omega_1)}^2.$$

The proof of this claim is well known and can be found e.g. in [Adm, p. 63 ff.].

The multiplication of a H^1 -function with a C^1 -function is again a H^1 -function and hence $(f \circ \eta^{-1})|\det D\eta^{-1}| \in H^1(\Omega_2)$ and $u \circ \eta$ is welldefined by (2.2). In fact we can estimate:

$$\begin{aligned}
|\langle u \circ \eta, f \rangle| &= |\langle u, (f \circ \eta^{-1})|\det D\eta^{-1}| \rangle| \\
&\leq \|u\|_{H^{-1}(\Omega_2)} \cdot \|(f \circ \eta^{-1})|\det D\eta^{-1}|\|_{H^1(\Omega_2)} \\
&\leq \text{const.} \|f \circ \eta^{-1}\|_{H^1(\Omega_2)} \leq \text{const.} \|f\|_{H^1(\Omega_1)}.
\end{aligned}$$

This completes the proof of Lemma 2.9. ■

Corollary 2.11

Suppose that η and η^{-1} in Lemma 2.9 are only C^1 -diffeomorphisms. Then: $u \circ \eta$, defined by (2.2), is an element of $H^{-1}(\Omega_1)$.

Proof:

For the proof of Claim 2.10 it was sufficient that $\eta, \eta^{-1} \in C^1$. It remains to

show the estimate for $|\langle u \circ \eta, f \rangle|$. We can estimate:

$$|\langle u \circ \eta, f \rangle| = \left| \int_{\Omega_2} u(f \circ \eta^{-1}) |\det D\eta^{-1}| dy \right| \leq \text{const.} \int_{\Omega_2} |u(f \circ \eta^{-1})| dy,$$

where the constant is $\max_{\overline{\Omega_2}} |\det D\eta^{-1}|$. The inequality

$$|\langle u, f \circ \eta^{-1} \rangle| \leq \text{const.} \|f\|_{H^1(\Omega_1)}$$

is given by Claim (2.10). Hence we obtain the inequality

$$|\langle u \circ \eta, f \rangle| \leq \text{const.} \|f\|_{H^1(\Omega_1)}.$$

■

Lemma 2.9 remains true, if η and η^{-1} are only Lipschitz continuous. This can be shown in the same way as Corollary 2.11 by using the following result which is due to Rademacher [Rad].

Lemma 2.12

Let $\eta: \Omega_1 \rightarrow \Omega_2$ be a Lipschitz continuous homeomorphism. Then:

- (i) $\frac{\partial \eta}{\partial x_i}$ exists a.e. in Ω_1 and all the partial derivatives are bounded.
- (ii) If $f \in L^1(\Omega_2)$, then $f \circ \eta \in L^1(\Omega_1)$ and

$$\int_{\Omega_2} f dy = \int_{\Omega_1} (f \circ \eta) |\det D\eta| dx.$$

Corollary 2.13

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^3$ be bounded domains and $\eta: \Omega_1 \rightarrow \Omega_2$ a transformation of class $C^{m,1}$. Furthermore let $u \in H^k(\Omega_2)$, $k \in \mathbb{Z}$, with $|k| \leq m+1$. Then: $u \circ \eta \in H^k(\Omega_1)$.

Corollary 2.14

Let Ω be of class $C^{0,1}$, then $X(\Omega) = L^2(\Omega)$.

The remainder of this chapter is similar to [Tar, p. 29-31]. Nevertheless we shall present it here to complete the argument for obtaining a result which will be of use for us later (see Lemma 2.19). The proof of Lemma 2.16 is not given in [Tar].

Lemma 2.15

Let Ω be of class $C^{0,1}$. Then the mapping

$$\text{grad} : L^2(\Omega) \longrightarrow (H^{-1}(\Omega))^3,$$

defined by

$$u \mapsto \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right)$$

has closed range in $(H^{-1}(\Omega))^3$.

For the proof we shall use the following

Lemma 2.16

Let E_1, E_2 and E_3 be Banach spaces. Furthermore let $A : E_1 \longrightarrow E_2$ be a bounded linear operator and $B : E_1 \longrightarrow E_3$ be a compact operator. Suppose that the norms

$$|||u|||_{E_1} \stackrel{\text{def}}{=} \|Au\|_{E_2} + \|Bu\|_{E_3}$$

and $\|u\|_{E_1}$ are equivalent on E_1 . Then:

- (i) $\dim(\ker A) < \infty$,
- (ii) $\text{range } A$ is closed in E_2 .

Proof:

(i)

Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\ker A \subset E_1$. Then - because of the compactness of B - $(Bu_n)_{n \in \mathbb{N}}$ contains a convergent subsequence in E_3 . Let us call this subsequence again $(Bu_n)_{n \in \mathbb{N}}$. Then $(u_n)_{n \in \mathbb{N}}$ is convergent in the $||| \cdot |||_{E_1}$ - norm since $|||u_n|||_{E_1} = \|Bu_n\|_{E_3}$. Hence $(u_n)_{n \in \mathbb{N}}$ is also convergent in the $\| \cdot \|_{E_1}$ - norm. Since $(u_n)_{n \in \mathbb{N}}$ was an arbitrary bounded sequence, $\dim(\ker A) < \infty$.

(ii)

By (i) there exists a subspace $X_0 \subset E$ such that $E_1 = \ker A \oplus X_0$. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in X_0 such that $(Au_n)_{n \in \mathbb{N}}$ is convergent in E_2 . Let us assume that $(Au_n)_{n \in \mathbb{N}}$ converges to $v \in E_2$.

Case 1: $(u_n)_{n \in \mathbb{N}}$ is bounded.

Then $(Bu_n)_{n \in \mathbb{N}}$ contains a convergent subsequence which we call again $(Bu_n)_{n \in \mathbb{N}}$. Since

$$\|u_n\|_{E_1} = \|Au_n\|_{E_2} + \|Bu_n\|_{E_3},$$

$(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in E_1 , and hence

$$u_n \xrightarrow{n \rightarrow \infty} u \in E_1,$$

therefore

$$Au_n \xrightarrow{n \rightarrow \infty} Au = v.$$

Case 2: $(u_n)_{n \in \mathbb{N}}$ is unbounded.

Define $x_n \stackrel{\text{def}}{=} \frac{u_n}{\|u_n\|_{E_1}}$. $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence and since $(Au_n)_{n \in \mathbb{N}}$ is convergent,

$$Ax_n = \frac{1}{\|u_n\|_{E_1}} Au_n \xrightarrow{n \rightarrow \infty} 0.$$

Furthermore $(Bx_n)_{n \in \mathbb{N}}$ contains a convergent subsequence which we call again $(Bx_n)_{n \in \mathbb{N}}$. This implies that $(x_n)_{n \in \mathbb{N}}$ is convergent in E_1 . Let us assume that $x_n \xrightarrow{n \rightarrow \infty} x_0 \in E_1$. Then

$$Ax_n \xrightarrow{n \rightarrow \infty} Ax_0 = 0$$

and $x_0 \in \ker A$. On the other hand $(u_n)_{n \in \mathbb{N}}$ belongs to X_0 , which implies that

$$x_0 \in X_0 \cap \ker A = \{0\}.$$

But this is a contradiction because of $\|x_n\|_{E_1} \equiv 1$, $\forall n \in \mathbb{N}$. Therefore Case 2 is impossible. ■

Proof of Lemma 2.15

The Banach spaces E_1 , E_2 and E_3 of Lemma 2.16 are now:

$$E_1 = L^2(\Omega), E_2 = (H^{-1}(\Omega))^3 \text{ and } E_3 = H^{-1}(\Omega).$$

Here the operators A and B are:

$$A = \text{grad} : L^2(\Omega) \rightarrow (H^{-1}(\Omega))^3$$

and

$$B = i : L^2(\Omega) \rightarrow H^{-1}(\Omega)$$

is the injection. We observe that

(i) i is a compact operator, since it is the adjoint of the compact imbedding $H_0^1(\Omega) \rightarrow L^2(\Omega)$.

(ii) In order to apply Lemma 2.16 we need to show that the norms $\|\cdot\|_{L^2}$ and

$$|||u|||_{L^2} \stackrel{\text{def}}{=} \|\text{grad } u\|_{(H^{-1})^3} + \|u\|_{H^{-1}}$$

are equivalent.

We first remark that grad and i are continuous operators and hence there exists a constant such that

$$\|\text{grad } u\|_{(H^{-1})^3} + \|u\|_{H^{-1}} \leq \text{const.} \|u\|_{L^2}.$$

This implies that every Cauchy sequence in $\|\cdot\|_{L^2}$ is already a Cauchy sequence in $|||\cdot|||_{L^2}$. Let $(u_n)_{n \in \mathbb{N}}$ now be a Cauchy sequence with respect to the $|||\cdot|||_{L^2}$. Then $(\text{grad } u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(H^{-1}(\Omega))^3$ and $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^{-1}(\Omega)$, i.e.,

$$\text{grad } u_n \xrightarrow{n \rightarrow \infty} v \in (H^{-1}(\Omega))^3$$

and

$$u_n \xrightarrow{n \rightarrow \infty} u \in H^{-1}(\Omega).$$

If $\varphi \in (\mathcal{D}(\Omega))^3$, then

$$\begin{aligned} \langle v, \varphi \rangle_{L^2} &= \lim_{n \rightarrow \infty} \langle \text{grad } u_n, \varphi \rangle_{L^2} \\ &= -\lim_{n \rightarrow \infty} \langle u_n, \text{div } \varphi \rangle_{L^2} \\ &= -\langle u, \text{div } \varphi \rangle_{L^2} = \langle \text{grad } u, \varphi \rangle_{L^2}. \end{aligned}$$

Since $\varphi \in (\mathcal{D}(\Omega))^3$ every term above is welldefined. Therefore we can conclude that

$$\text{grad } u = v \in (H^{-1}(\Omega))^3.$$

Hence $u \in X(\Omega) = L^2(\Omega)$ by Corollary 2.14, which implies that

$$u_n \xrightarrow{n \rightarrow \infty} u \text{ in } L^2(\Omega).$$

■

Lemma 2.17

Let $\Omega \subset \mathbb{R}^3$ be of class $C^{0,1}$ and $f \in (H^{-1}(\Omega))^3$. Furthermore suppose that $\langle f, u \rangle = 0$, $\forall u \in (H_0^1(\Omega))^3$ with $\text{div } u = 0$. Then $\exists p \in L^2(\Omega)$ with $f = \text{grad } p$.

Proof:

Define the following subsets of $(H^{-1}(\Omega))^3$ and $(H_0^1(\Omega))^3$, respectively:

$$\begin{aligned} Y &\stackrel{\text{def}}{=} \text{range}(\text{grad}) = \{f \in (H^{-1}(\Omega))^3 : f = \text{grad } p, p \in L^2(\Omega)\}, \\ Z &\stackrel{\text{def}}{=} \{u \in (H_0^1(\Omega))^3 : \text{div } u = 0\}. \end{aligned}$$

Claim 2.18

$$Y^\perp = Z,$$

where $Y^\perp$ is the orthogonal complement in the duality $\langle \cdot, \cdot \rangle$ (see Def. 2.4).

Proof:

$$u \in Y^\perp \iff \langle u, \text{grad } p \rangle = 0, \forall p \in L^2(\Omega).$$

For $p \in \mathcal{D}(\Omega)$ we can compute:

$$\langle u, \text{grad } p \rangle = -\langle \text{div } u, p \rangle = 0, \forall p \in \mathcal{D}(\Omega) \iff \text{div } u = 0.$$

Then we can write:

$$Z^\perp = Y^{\perp\perp} = \overline{Y} = Y.$$

The last equality holds because of Lemma 2.15. Writing this out leads to:

$$\begin{aligned} f \in Z^\perp &\iff f \in (H^{-1}(\Omega))^3 \text{ with } \langle f, u \rangle = 0, \forall u \in Z \\ &\iff f \in Y \\ &\iff \exists p \in L^2(\Omega) \text{ with } f = \text{grad } p. \end{aligned}$$

Lemma 2.19

Let Ω be a domain of class $C^{0,1}$ and $f \in (H^{-1}(\Omega))^3$ with

$$\langle f, \varphi \rangle = 0, \forall \varphi \in \mathcal{V} = \{\varphi \in (\mathcal{D}(\Omega))^3 : \text{div } \varphi = 0\}.$$

Then $\exists p \in L^2(\Omega)$ with $f = \text{grad } p$.

Proof:

Case 1: Ω is strictly star shaped with respect to some point x_0 .

Without loss of generality we can assume that $x_0 = 0$. Given $0 \leq \theta < 1$, $\theta\bar{\Omega} \subset \Omega$. We then define for an $u \in (H_0^{-1}(\Omega))^3$ with $\operatorname{div} u = 0$: $u_\theta \stackrel{\text{def}}{=} u \left(\frac{x}{\theta} \right)$.

Hence

$$\operatorname{div} u_\theta = \theta^{-1} \operatorname{div} u = 0,$$

u_θ has compact support in Ω and u_θ can be approximated by elements of $(\mathcal{D}(\Omega))^3$. Let $\rho \in (\mathcal{D}(\Omega))^3$, $\rho \geq 0$ and

(i) $\rho = 0$, if $|x| \geq 1$,

(ii) $\int_{\mathbf{R}^3} \rho \, dx = 1$.

Let us define

$$\rho_\epsilon(x) \stackrel{\text{def}}{=} \frac{1}{\epsilon^3} \rho \left(\frac{x}{\epsilon} \right).$$

Moreover the convolution of ρ with u_θ , $\rho * u_\theta$, is an element in $(\mathcal{D}(\Omega))^3$ for ϵ small enough and $\rho * u_\theta \xrightarrow{\epsilon \rightarrow 0} u_\theta$. We compute that

$$\langle f, u_\theta \rangle = \lim_{\epsilon \rightarrow 0} \langle f, \rho_\epsilon * u_\theta \rangle = 0,$$

since $\langle f, \rho_\epsilon * u_\theta \rangle = 0$ by our assumption on f . From this it follows that

$$\langle f, u \rangle = \lim_{\theta \rightarrow 1} \langle f, u_\theta \rangle = 0, \forall u \in (H_0^1(\Omega))^3 \text{ with } \operatorname{div} u = 0.$$

Using Lemma 2.17 we can conclude that $\exists p \in L^2(\Omega)$ with $f = \operatorname{grad} p$.

Case 2: Ω is not star shaped.

Each point of $\partial\Omega$ has a neighbourhood ω which is of class $C^{0,1}$ and is strictly star shaped. Then $\exists q \in L^2(\omega)$ with $f = \operatorname{grad} q$ in ω .

Since $q_1|_{\omega_1 \cap \omega_2} - q_2|_{\omega_1 \cap \omega_2} = \text{const.}$, $\exists p \in L^2(\Omega)$ with $p|_\omega - q = \text{const.}$ and $f = \operatorname{grad} p$ in Ω . ■

2.2 Equivalent norms for the spaces V and J

Let us first recall the definition of the space V . V is the closure of $\dot{V} \subset (H^1(\Omega))^3$ in the $\|\cdot\|_{H^1}$ (see Def. 2.1). $\dot{V}$ is defined by

$$\begin{aligned} \dot{V} \stackrel{\text{def}}{=} \{v \in (C^\infty(\Omega))^3 : & \partial^\alpha v|_{x=0} = \partial^\alpha v|_{x=x_0}, \partial^\alpha v|_{y=0} = \partial^\alpha v|_{y=1}, \\ & \forall |\alpha| \leq 1; \ v|_{\Gamma_b} = 0; \ v \cdot n|_\Gamma = 0\}. \end{aligned}$$

In the following we define two inner products, one on V and one on J , which induce equivalent norms with respect to the $\|\cdot\|_{H^1}$. We are following the arguments outlined in [Sol/Sca] where the corresponding results have been proven for the space

$$V_S = cl \left(\{v \in (C^\infty(\Omega))^3 : v|_{\Gamma_1} = 0; v \cdot n|_{\Gamma_2} = 0\} \right).$$

In [Sol/Sca] the closure is taken with respect to the $\|\cdot\|_{H^1}$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$.

Definition 2.20

If $u, v \in (H^1(\Omega))^3$, then

$$\langle \nabla u, \nabla v \rangle_{L^2} = \sum_{i,j=1}^3 \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx$$

induces a seminorm on $(H^1(\Omega))^3$ which is called the gradient norm $\|\cdot\|_{\nabla}$,

$$\|\nabla u\|_{L^2}^2 = \|u\|_{\nabla}^2 = \sum_{i,j=1}^3 \int_{\Omega} \left| \frac{\partial u_i}{\partial x_j} \right|^2 dx.$$

Lemma 2.21

$\|\cdot\|_{\nabla}$ defines a norm on V . Moreover $\|\cdot\|_{\nabla}$ is equivalent to $\|\cdot\|_{H^1}$.

Proof:

Since

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2,$$

it is sufficient to show that the Poincaré inequality is true for $u \in V$, i.e.: $\exists c_0$ such that

$$\|u\|_{L^2}^2 \leq c_0 \cdot \|\nabla u\|_{L^2}^2, \forall u \in V.$$

Let $x = (x_1, x_2, x_3) \in \Omega$ and $x' = (x_1, x_2)$. The shape of the domain Ω is such that each point $x = (x', x_3)$ can be reached from the bottom Γ_b by starting at $x_{\Gamma_b} = (x', 0)$ and going along the vertical up to $x = (x', x_3)$. This path lies entirely in Ω . We can then compute $u(x)$ by integrating over the vertical line, i.e.,

$$u(x', x_3) = \int_0^{x_3} \frac{\partial u(x', t)}{\partial t} dt + \underbrace{u(x', 0)}_{=0 \text{ for } u \in V}.$$

Squaring this equality and using Hölder's inequality leads to

$$\begin{aligned}
(u(x', x_3))^2 &= \left(\int_0^{x_3} \frac{\partial u(x', t)}{\partial t} dt \right)^2 \\
&\leq \left[\left(\int_0^{x_3} dt \right)^{1/2} \cdot \left(\int_0^{x_3} \left(\frac{\partial u(x', t)}{\partial t} \right)^2 dt \right)^{1/2} \right]^2 \\
&= x_3 \cdot \int_0^{x_3} \left(\frac{\partial u(x', t)}{\partial t} \right)^2 dt.
\end{aligned}$$

We integrate u^2 over Ω and obtain

$$\begin{aligned}
\int_{\Omega} u^2 dx &\leq \int_{x'} \int_{x_3} x_3 \cdot \int_0^{x_3} \left(\frac{\partial u(x', t)}{\partial t} \right)^2 dt dx_3 dx' \\
&\leq \int_{x'} \int_{x_3} x_3 \cdot \int_0^{l_3(x')} \left(\frac{\partial u(x', t)}{\partial t} \right)^2 dt dx_3 dx' \\
&\leq \frac{L^2}{2} \int_0^{l_3(x')} \left(\frac{\partial u}{\partial t} \right)^2 dt dx',
\end{aligned}$$

where $l_3(x')$ is such that $(x', l_3(x')) \in \Gamma$ and $L \geq \sup_{x'} l_3(x')$. This leads to the assertion

$$\int_{\Omega} u^2 dx \leq \frac{L^2}{2} \int_{\Omega} \left(\frac{\partial u}{\partial x_3} \right)^2 dx \leq \frac{L^2}{2} \int_{\Omega} (\nabla u)^2 dx = \frac{L^2}{2} \|\nabla u\|_{L^2}^2.$$

■

Next we shall prove that the bilinear mapping $E(\cdot, \cdot)$ which is given below defines an inner product on J .

Definition 2.22

$$E(u, v) \stackrel{def}{=} \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) dx.$$

Remark 2.23

Let $u \in V$ and $E(u, u) = 0$, then $u = 0$.

Proof:

It is known (see e.g. [Sol/Sca]) that $E(u, u) = 0$ iff $u = A + B \times x$ with constant vectors A and B . If $B = (b_1, b_2, b_3)$, then

$$B \times x = \begin{pmatrix} b_2 x_3 - b_3 x_2 \\ b_3 x_1 - b_1 x_3 \\ b_1 x_2 - b_2 x_1 \end{pmatrix}.$$

The periodicity at $x = 0$ and $x = x_0$ implies that

$$B \times x|_{x=0} = \begin{pmatrix} b_2 x_3 - b_3 x_2 \\ -b_1 x_3 \\ b_1 x_2 \end{pmatrix} = B \times x|_{x=x_0} = \begin{pmatrix} b_2 x_3 - b_3 x_2 \\ b_3 x_0 - b_1 x_3 \\ b_1 x_2 - b_2 x_0 \end{pmatrix}, \forall x_2, x_3.$$

If $x_3 = 0$, then $b_3 = 0$. If we set $x_2 = 0$, then $b_2 = 0$. The periodicity at $x_2 = 0$ and $x_2 = x_0$ finally leads to $b_1 = 0$. Since $B = 0$ and $u|_{\Gamma_b} = 0$, it follows that $A = 0$.

Corollary 2.24

$E(u, v)$ defines a scalar product on V and the induced norm

$$\|u\|_E \stackrel{\text{def}}{=} E(u, u)$$

is called the energy norm.

Lemma 2.25

For all $v \in J$ there exists a constant c such that

$$\|v\|_{\nabla}^2 \leq c \cdot \|v\|_E^2.$$

Proof:

We compute

$$\begin{aligned} E(v, v) &= \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)^2 dx \\ &= \underbrace{2 \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial v_i}{\partial x_k} \right)^2 dx}_{= \|v\|_{\nabla}^2} + 2 \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_i} \right)^2 dx. \quad (2.4) \end{aligned}$$

The second term can be written as

$$\begin{aligned} \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_i} \right)^2 dx &= - \sum_{i,k=1}^3 \int_{\Omega} v_i \frac{\partial}{\partial x_k} \frac{\partial v_k}{\partial x_i} dx + \sum_{i,k=1}^3 \int_{\partial\Omega} v_i n_k \frac{\partial v_k}{\partial x_i} ds \\ &= - \sum_{i=1}^3 \int_{\Omega} v_i \frac{\partial}{\partial x_i} \underbrace{\left(\sum_{k=1}^3 \frac{\partial v_k}{\partial x_k} \right)}_{\operatorname{div} v = 0} dx + \sum_{i,k=1}^3 \int_{\partial\Omega} v_i n_k \frac{\partial v_k}{\partial x_i} ds, \end{aligned}$$

where we have used that $\operatorname{div} v = 0$ for $v \in J$. The boundary integral is only an integral over Γ because of the boundary conditions for $v \in J$. This can be seen in the following way:

$$\begin{aligned} \sum_{i=1}^3 \int_{\partial\Omega} v_i n_k \frac{\partial v_k}{\partial x_i} ds &= \int_{\partial\Omega} v_i n_k \frac{\partial v_k}{\partial x_i} ds \\ &= \underbrace{\int_{\Gamma_x} v_i n_k \frac{\partial v_k}{\partial x_i} ds}_{= 0 \text{ since } \partial_i v_k|_{x=0} = \partial_i v_k|_{x=x_0}} + \underbrace{\int_{\Gamma_y} v_i n_k \frac{\partial v_k}{\partial x_i} ds}_{= 0 \text{ since } \partial_i v_k|_{y=0} = \partial_i v_k|_{y=1}} + \\ &\quad + \underbrace{\int_{\Gamma_b} v_i n_k \frac{\partial v_k}{\partial x_i} ds}_{= 0 \text{ since } v|_{\Gamma_b} = 0} + \int_{\Gamma} v_i n_k \frac{\partial v_k}{\partial x_i} ds, \end{aligned}$$

where

$$\Gamma_x = \partial\Omega \cap (\{x = 0\} \cup \{x = x_0\})$$

and

$$\Gamma_y = \partial\Omega \cap (\{y = 0\} \cup \{y = 1\}).$$

So we can write:

$$\sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_i} \right)^2 dx = \sum_{i,k=1}^3 \int_{\Gamma} v_i n_k \frac{\partial v_k}{\partial x_i} ds. \quad (2.5)$$

In order to show the equivalence of the norms we need to estimate the boundary integral in (2.5) for $v \in (C^2(\Omega))^3 \cap J$. We can assume that Γ is as

smooth as we please. So let us consider Γ as a C^2 -manifold. Let $\sigma \subset \Gamma$ be open and the zero set of $G(x) \stackrel{\text{def}}{=} x_3 - F(x_1, x_2)$ where F is a C^2 -function. Then we can express the k^{th} -component of the unit normal vector n_k in terms of G or F , respectively.

$$n_k(x) = \frac{1}{\sqrt{|\nabla G|^2}} \frac{\partial G}{\partial x_k} = \frac{1}{\sqrt{1 + |\nabla F|^2}} \cdot \begin{cases} \frac{\partial F}{\partial x_k} & \text{for } k = 1, 2, \\ 1 & \text{for } k = 3. \end{cases}$$

We shall show now that

$$\sum_{i,k=1}^3 \int_{\sigma} v_i n_k \frac{\partial v_k}{\partial x_i} ds$$

can be estimated by $\|v\|_{L^2(\sigma)}^2$.

Claim 2.26

Let $v \in (C^2(\Omega_n))^3 \cap J$. Then $\exists$ a positive constant c_0 such that

$$\left| \sum_{i,k=1}^3 \int_{\Gamma} v_i n_k \frac{\partial v_k}{\partial x_i} ds \right| \leq c_0 \int_{\Gamma} v^2 ds.$$

The constant c_0 depends only on the function G .

Remark:

(i) In the claim we have assumed that $\sigma = \Gamma$. If this is not the case, then a partition of unity argument has to be used in addition.

(ii) The space $(C^2(\Omega_n))^3$ is defined as the space of functions which are C^2 in an open neighbourhood of Ω . $(C^2(\Omega_n))^3$ is dense in $(H^1(\Omega))^3$ (see e.g. [Adm, Th. 3.18]).

Proof:

The integral over Γ can be transformed into an integral over Γ_0 . Let (t_1, t_2) be the coordinates on Γ_0 . Then the mapping $\Phi : \Gamma_0 \rightarrow \Gamma$ given by

$$(t_1, t_2) \mapsto (t_1, t_2, F(t_1, t_2))$$

defines a chart for Γ . For the transformed integral we obtain (see e.g. [For, p. 245 ff.]):

$$\begin{aligned} I(\Gamma) &= \sum_{i,k=1}^3 \int_{\Gamma} v_i n_k \frac{\partial v_k}{\partial x_i} ds \\ &= - \sum_{k=1}^2 \sum_{i=1}^3 \int_{\Gamma_0} v_i(\Phi(t)) \frac{\partial v_k}{\partial x_i} \frac{\partial F}{\partial t_k} dt + \sum_{i=1}^3 \int_{\Gamma_0} v_i(\Phi(t)) \frac{\partial v_3}{\partial x_i}(\Phi(t)) dt. \end{aligned} \quad (2.6)$$

Define $V_k \stackrel{\text{def}}{=} v_k(\Phi(t))$. Then the partial derivatives can be computed as

$$\frac{\partial V_k}{\partial t_i} = \sum_{s=1}^3 \frac{\partial v_k}{\partial x_s}(\Phi(t)) \frac{\partial \Phi_s}{\partial t_i} = \frac{\partial v_k}{\partial x_i}(\Phi(t)) + \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_i} \quad (2.7)$$

or

$$\frac{\partial v_k}{\partial x_i}(\Phi(t)) = \frac{\partial V_k}{\partial t_i} - \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_i} \quad (i = 1, 2), (k = 1, 2, 3).$$

Substituting this expression into (2.6) leads to

$$\begin{aligned} I(\Gamma) &= - \sum_{i,k=1}^2 \int_{\Gamma_0} V_i \frac{\partial V_k}{\partial t_i} \frac{\partial F}{\partial t_k} dt + \sum_{i,k=1}^2 \int_{\Gamma_0} V_i \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_i} \frac{\partial F}{\partial t_k} dt - \\ &\quad \sum_{k=1}^2 \int_{\Gamma_0} V_3 \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_k} dt + \sum_{i=1}^2 \int_{\Gamma_0} V_i \frac{\partial V_3}{\partial t_i} dt - \\ &\quad \sum_{i=1}^2 \int_{\Gamma_0} V_i \frac{\partial v_3}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_i} dt + \int_{\Gamma_0} V_3 \frac{\partial v_3}{\partial x_3}(\Phi(t)) dt. \end{aligned} \quad (2.8)$$

Since $v \cdot n|_{\Gamma} = 0$ is equivalent to

$$V \cdot \nabla G = -V_1 \frac{\partial F}{\partial t_1} - V_2 \frac{\partial F}{\partial t_2} + V_3 = 0$$

it follows that

$$\begin{aligned} &\sum_{i,k=1}^2 \int_{\Gamma_0} V_i \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_i} \frac{\partial F}{\partial t_k} dt - \sum_{k=1}^2 \int_{\Gamma_0} V_3 \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_k} dt \\ &= \sum_{k=1}^3 \int_{\Gamma_0} \underbrace{\left(V_1 \frac{\partial F}{\partial t_1} + V_2 \frac{\partial F}{\partial t_2} - V_3 \right)}_{=0} \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_k} ds = 0 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^2 \int_{\Gamma_0} V_i \frac{\partial v_3}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_i} dt + \int_{\Gamma_0} V_3 \frac{\partial v_3}{\partial x_3}(\Phi(t)) dt &= \\ = \int_{\Gamma_0} \left(-V_1 \frac{\partial F}{\partial t_1} - V_2 \frac{\partial F}{\partial t_2} + V_3 \right) \frac{\partial v_3}{\partial x_3}(\Phi(t)) dt &= 0. \end{aligned}$$

Hence (2.8) reduces to two integrals. Applying Gauß' theorem to those integrals leads to:

$$I(\Gamma) = \sum_{i,k=1}^2 \int_{\Gamma_0} V_k V_i \frac{\partial^2 F}{\partial t_i \partial t_k} dt - \sum_{i=1}^2 \int_{\Gamma_0} V_3 \frac{\partial V_i}{\partial t_i} dt. \quad (2.9)$$

Since $v \in (C^2(\Omega_n))^3$, $\operatorname{div} v$ is continuous in a neighbourhood of Ω . This implies that $\operatorname{div} v|_{\Gamma} = 0$. Using (2.7) $\operatorname{div} v|_{\Gamma}$ can be written as

$$0 = (\operatorname{div} v)(\Phi(t)) = \sum_{k=1}^3 \frac{\partial v_k}{\partial x_k}(\Phi(t)) = \sum_{k=1}^2 \frac{\partial V_k}{\partial t_k} - \sum_{k=1}^3 \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_k} + \frac{\partial v_3}{\partial x_3}(\Phi(t)).$$

Furthermore the condition $v \cdot n|_{\Gamma} = 0$ can be used in the form

$$\begin{aligned} 0 = \left(\frac{\partial}{\partial x_3} (v \cdot \nabla G) \right) (\Phi(t)) &= \left(\frac{\partial v}{\partial x_3} \cdot \nabla G \right) (\Phi(t)) + \left(v \cdot \frac{\partial (\nabla G)}{\partial x_3} \right) (\Phi(t)) \\ &= - \sum_{k=1}^2 \frac{\partial v_k}{\partial x_3}(\Phi(t)) \frac{\partial F}{\partial t_k} + \frac{\partial v_3}{\partial x_3}(\Phi(t)), \end{aligned}$$

since $\partial_3(\nabla G) = 0$. Using the last two equations we obtain $\sum_{k=1}^2 \partial V_k / \partial t_k = 0$. The result in (2.9) finally proofs the claim, since

$$I(\Gamma) = - \sum_{i,k=1}^2 \int_{\Gamma_0} V_i V_k \frac{\partial^2 F}{\partial t_i \partial t_k} dt$$

which can be estimated by

$$|I(\Gamma)| \leq \operatorname{const.} \int_{\Gamma_0} V^2 dt.$$

The constant in this estimate depends only on F .

Claim 2.27

Let $v \in (C^2(\Omega))^3 \cap J$ and $\delta > 0$ be given. Then $\exists M > 0$ such that

$$\|v\|_{L^2(\Gamma)}^2 \leq \delta \|v\|_{\nabla}^2 + M \|v\|_E^2.$$

Using Claim 2.26 and Claim 2.27 we are now able to finish the proof of Lemma 2.25. $I(\Gamma)$ can then be estimated by

$$\left| \sum_{i,k=1}^3 \int_{\Gamma} v_i n_k \frac{\partial v_k}{\partial x_i} ds \right| \leq \frac{1}{2} \|v\|_{\nabla}^2 + M_0 \|V\|_E^2.$$

Here we have chosen δ and M from Claim 2.27 to be $\delta = \frac{1}{2c_0}$ and $M = M/c_0$. Combining this estimate with (2.4) and (2.5) we obtain

$$\|v\|_{\nabla}^2 \leq \frac{1}{2} \|v\|_E^2 + \left| \sum_{i,k=1}^3 \int_{\Gamma} v_i n_k \frac{\partial v_k}{\partial x_i} ds \right| \leq \frac{1}{2} \|v\|_{\nabla}^2 + (M_0 + \frac{1}{2}) \|v\|_E^2$$

or

$$\frac{1}{2} \|v\|_{\nabla}^2 \leq (2M_0 + 1) \|v\|_E^2, \quad \forall v \in (C^2(\Omega))^3 \cap J.$$

■

It remains to prove Claim 2.27.

Proof of Claim 2.27

Let us assume that for a given $\delta_0 > 0$ there is no $M_0 > 0$ for which the assertion is true. Then $\forall m \in \mathbb{N} \exists v_m \in (C^2(\Omega))^3 \cap J$ such that

$$\|v_m\|_{L^2(\Gamma)}^2 \geq \delta_0 \|v_m\|_{\nabla}^2 + m \|v_m\|_E^2.$$

Define

$$w_m \stackrel{def}{=} \frac{v_m}{\|v_m\|_{L^2(\Gamma)}}.$$

Then

$$\|w_m\|_{L^2(\Gamma)}^2 = 1 \geq \delta_0 \|w_m\|_{\nabla}^2 + m \|w_m\|_E^2.$$

Since both terms on the right hand side are positive, each of them is bounded by 1.

$$(i) \quad \|w_m\|_{\nabla}^2 \leq 1.$$

This implies that there exists a weakly convergent subsequence of

$$(w_m)_{m \in \mathbb{N}} \in H^1(\Omega) \cap J$$

which we again call $(w_m)_{m \in \mathbb{N}}$. Let us call w the weak limit

$$w_m \xrightarrow{weakly} w \in H^1(\Omega).$$

Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ is compact,

$$w_m \xrightarrow{\text{strongly}} w \in L^2(\Gamma).$$

Hence we conclude that

$$\|w\|_{L^2(\Gamma)} = \lim_{m \rightarrow \infty} \|w_m\|_{L^2(\Gamma)} = 1.$$

(ii) $m\|w_m\|_E \leq 1$.

Since $\|\cdot\|_E = E(\cdot, \cdot)$ is a quadratic form, the inequality

$$E(w_m, w_m) + E(w, w) \geq 2 E(w_m, w)$$

holds. $E(\cdot, w)$ is a lower semicontinuous function and therefore

$$\liminf_{m \rightarrow \infty} (E(w_m, w_m) + E(w, w)) \geq 2 \liminf_{m \rightarrow \infty} E(w_m, w) \geq 2 E(w, w).$$

From this we conclude that $E(w, w) \leq 0$ or $E(w, w) = 0$, which implies $w = 0$ by Remark 2.23. This however is impossible, since $\|w\|_{L^2(\Gamma)} = 1$.

Corollary 2.28

$E(\cdot, \cdot)$ defines a scalar product on J . The topology induced by this scalar product is equivalent to the norm induced by the gradient norm $\|\cdot\|_{\nabla}$.

Proof:

All we need to show is that $\exists$ a constant c_0 such that

$$E(v, v) \leq c_0 \|v\|_{\nabla}^2.$$

A straightforward computation shows that we can choose $c_0 = 4$. ■

2.3 The weak formulation

In this part we shall define what we mean by a weak solution of our problem. This will be done for the linearized equation. We continue to follow the lines which are given in [Sol/Sca]. The periodic boundary conditions for p in our problem will turn out to be the right choice. That means that it is not only sufficient for transforming the boundary value problem into the weak formulation but that those conditions are also necessary when we transform the weak formulation back into a boundary value problem. The proof for this can be carried out in essentially the same way as showing that the tangential stress condition is satisfied on the free surface.

2.3.1 The linearized problem

Throughout this part we shall always assume that the domain Ω is known or is given by some function ζ . Therefore we assume that the normal stress condition is satisfied and we do no longer have to deal with a free boundary during this part. The remaining boundary value problem can be transformed into an equivalent integral equation. In order to show this equivalence it is of course not necessary to carry the nonlinearity all the way through the computations. As a matter of fact it is sufficient to show the equivalence for the Stokes Problem and then carry the result over to the problem that we want to consider.

The linearized problem with zero boundary conditions on Γ_b is described by the following set of equations (see (1.53)):

$$\left. \begin{aligned} -\nu \Delta u + \text{grad } p + u_{0k} \partial_k u + u_k \partial_k u_0 + u_k \partial_k u &= f \\ \text{div } u &= 0 \end{aligned} \right\} \quad \text{in } \Omega \quad (2.10)$$

$$\left. \begin{aligned} \partial^\alpha u|_{x=0} &= \partial^\alpha u|_{x=x_0} \\ \partial^\alpha u|_{y=0} &= \partial^\alpha u|_{y=1} \\ p|_{x=0} &= p|_{x=x_0} \\ p|_{y=0} &= p|_{y=1} \\ u|_{\Gamma_b} &= 0 \quad (B2) \\ u \cdot n|_{\Gamma} &= 0 \quad (B3) \\ \tau_k \cdot t(u) \cdot n|_{\Gamma} &= \gamma_k, \quad k = 1, 2. \end{aligned} \right\} \quad |\alpha| \leq 1 \quad (B1) \quad (2.11)$$

As we said before it is sufficient to look at the Stokes problem, i.e.,

$$\left. \begin{aligned} -\nu \Delta u + \text{grad } p &= f \\ \text{div } u &= 0. \end{aligned} \right\} \quad \text{in } \Omega. \quad (2.12)$$

The boundary conditions are the same as in (2.11).

Let us take a function $\varphi \in J$, multiply (2.12) with φ and integrate over Ω . Then we obtain

$$\langle -\nu \Delta u + \text{grad } p, \varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2}. \quad (2.13)$$

For the upcoming computations we assume that $u, \varphi \in (C^2(\Omega))^3 \cap J$ which is a dense set in J .

Computations for the stress tensor

The stress tensor $t = (t_{ik})$ is defined in Cartesian coordinates by

$$t_{ik} = -p\delta_{ik} + \nu(\partial_k u_i + \partial_i u_k).$$

The divergence of t can be computed as

$$\operatorname{div} t = \sum_k \partial_k t_{ik} = - \sum_k \partial_k p \delta_{ik} + \nu \sum_k (\partial_k^2 u_i + \partial_{ik}^2 u_k).$$

The last sum above is zero, since

$$\sum_k \partial_{ik}^2 u_k = \partial_i \left(\sum_k \partial_k u_k \right) = \partial_i (\operatorname{div} u)$$

and $\operatorname{div} u = 0$. Therefore the divergence of t can be written as

$$-\operatorname{div} t = -\nu \Delta u + \operatorname{grad} p. \quad (2.14)$$

Furthermore we can compute that

$$\begin{aligned} \nabla(t \cdot \varphi) &= \underbrace{(\operatorname{div} t) \cdot \varphi}_{\sum_{i,k} \partial_k t_{ik}} + \underbrace{t \cdot \operatorname{grad} \varphi}_{\sum_{i,k} t_{ik} \partial_k \varphi_i}. \end{aligned}$$

Using this equation and (2.14) the left hand side of (2.13) can be written as:

$$\langle -\nu \Delta u + \operatorname{grad} p, \varphi \rangle_{L^2} = - \int_{\Omega} \nabla(t \cdot \varphi) dx + \langle t, \operatorname{grad} \varphi \rangle_{L^2}. \quad (2.15)$$

When we apply Gauß' theorem to the first integral on the right hand side we obtain the boundary integral

$$\begin{aligned} \int_{\Omega} \nabla(t \cdot \varphi) dx &= \int_{\partial\Omega} t \cdot \varphi \cdot n ds = \sum_{i,k} \int_{\partial\Omega} t_{ik} \varphi_k n_i ds \\ &= \sum_{i,k} \int_{\partial\Omega} (-p) \underbrace{\delta_{ik} \varphi_k n_i}_{\varphi \cdot n|_{\partial\Omega}} ds + \nu \sum_{i,k} \int_{\partial\Omega} (\partial_k u_i + \partial_i u_k) \varphi_k n_i ds. \end{aligned}$$

Because of the periodicity of p , u and φ both of the integrals on the right hand side are zero when taken only over Γ_x and Γ_y (see also the proof of

Lemma 2.25). The integrals over Γ_b are zero because $\varphi|_{\Gamma_b} = 0$. Hence there are only integrations over Γ left, i.e.,

$$\int_{\Omega} \nabla(t \cdot \varphi) dx = \int_{\Gamma} \varphi \cdot (t \cdot n) ds. \quad (2.16)$$

Since $\varphi \cdot n|_{\Gamma} = 0$, $\varphi|_{\Gamma}$ is at each point $x \in \Gamma$ a vector in the tangent space of Γ . Let us assume that τ_1 and τ_2 form an orthonormal basis of $T_x\Gamma$. Then φ can be written as $\varphi|_{\Gamma} = \alpha\tau_1 + \beta\tau_2$ and the boundary condition for the tangential stress in (2.11) becomes

$$\varphi \cdot t \cdot n|_{\Gamma} = \alpha\gamma_1 + \beta\gamma_2,$$

where α and β are functions on Γ with values in $\mathbf{R}$. Using this result and (2.16), equation (2.15) becomes

$$\langle -\nu\Delta u + \text{grad } p, \varphi \rangle_{L^2} = \langle t, \text{grad } \varphi \rangle_{L^2} - \int_{\Gamma} (\alpha\gamma_1 + \beta\gamma_2) ds. \quad (2.17)$$

The term $\langle t, \text{grad } \varphi \rangle_{L^2}$ can be computed further as

$$\langle t, \text{grad } \varphi \rangle_{L^2} = - \sum_{i,k} \int_{\Omega} p \delta_{ik} \partial_k \varphi_i dx + \nu \sum_{i,k} \int_{\Omega} (\partial_k u_i + \partial_i u_k) \partial_k \varphi_i dx,$$

where the first integral is zero because of

$$\sum_{i,k} p \delta_{ik} \partial_k \varphi_i = p \sum_k \partial_k \varphi_k = p(\text{div } \varphi) = 0.$$

We observe that i and k can be interchanged in the second integral without changing the integral. Hence

$$\begin{aligned} \langle t, \text{grad } \varphi \rangle_{L^2} &= \frac{\nu}{2} \sum_{i,k=1}^3 \int_{\Omega} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial \varphi_i}{\partial x_k} + \frac{\partial \varphi_k}{\partial x_i} \right) dx \\ &= \frac{\nu}{2} E(u, \varphi). \end{aligned} \quad (2.18)$$

We can summarize our computations in the following Lemma.

Lemma 2.29

If $u \in J$ is a solution of (2.12) in the distributional sense, then u satisfies the integral equation

$$\frac{\nu}{2} E(u, \varphi) = \langle f, \varphi \rangle_{L^2} + \int_{\Gamma} ((\varphi \cdot \tau_1)\gamma_1 + (\varphi \cdot \tau_2)\gamma_2) ds, \quad \forall \varphi \in J. \quad (2.19)$$

Definition 2.30

u is called a weak solution of (2.12) whenever u satisfies the integral equation (2.19).

Naturally the question arises, if a weak solution is also a solution of the boundary value problem in the distributional sense. The answer is affirmative.

Theorem 2.31

Assume that $f \in (L^2(\Omega))^3$ and $u \in J$ is a weak solution of (2.12). Then $\exists p \in L^2(\Omega)$ such that (u, p) is a solution of (2.12) in the distributional sense. Moreover if $u \in (H^2(\Omega))^3$ and $p \in H^1(\Omega)$, then (u, p) satisfies also the boundary conditions a.e..

Proof:

Take

$$\varphi \in \mathcal{V} = \{\psi \in (\mathcal{D}(\Omega))^3 : \operatorname{div} \psi = 0\}$$

and substitute it for φ in (2.19). This leads to

$$\frac{\nu}{2} E(u, \varphi) = \langle f, \varphi \rangle_{L^2}$$

and

$$\begin{aligned} \frac{\nu}{2} E(u, \varphi) &= \nu \sum_{i,k=1}^3 \int_{\Omega} (\partial_k u_i + \partial_i u_k) \partial_k \varphi_i \, dx \\ &= -\nu \left\{ \sum_{i,k=1}^3 \int_{\Omega} \partial_{kk}^2 u_i \varphi_i \, dx + \sum_{i,k=1}^3 \int_{\Omega} \partial_{ik}^2 u_k \varphi_i \, dx \right\}. \end{aligned}$$

In the computation above we have applied Gauß' theorem. The boundary integral is zero, since $\varphi|_{\partial\Omega} = 0$. The second integral on the right hand side is also zero, since

$$\sum_{i,k} \partial_{i,k}^2 u_k = \partial_i (\operatorname{div} u) = 0.$$

Hence we have shown that

$$(\nu \Delta u + f, \varphi)_{L^2} = 0, \quad \forall \varphi \in \mathcal{V}. \quad (2.20)$$

Then by Lemma 2.19 $\exists p \in L^2(\Omega)$ such that

$$\nu \Delta u + f = \operatorname{grad} p$$

or

$$-\nu \Delta u + \operatorname{grad} p = f \quad (2.21)$$

$$\operatorname{div} u = 0. \quad (2.22)$$

This shows that $u \in (H^1(\Omega))^3$ and $p \in L^2(\Omega)$ solve the Stokes problem (2.12). The periodic boundary conditions for u are satisfied by definition of the space J whenever u is smooth enough such that the trace is defined on the boundary. It remains to show that the boundary conditions of the tangential stress equation and the periodicity condition for p are also satisfied. To show this we shall need the following lemma which is due to Ladyzhenskaya [Lad, p. 24 ff.].

Lemma 2.32

Let $a \in (H^{1/2}(\partial\Omega))^3$ and let Ω be a domain of class C^2 . Furthermore suppose that

$$\int_{\partial\Omega} a \cdot n \, ds = 0.$$

Then $\exists \varphi \in (H^1(\Omega))^3$ such that $\operatorname{div} \varphi = 0$ and $\varphi|_{\partial\Omega} = a$.

In the following we shall assume that $p \in H^1(\Omega)$ and $u \in (H^2(\Omega))^3$ so that the traces of $t(u)$ and p are defined on Γ , Γ_x and Γ_y .

Boundary condition for the tangential stress

Each inner point of Γ lies in a C^2 -neighbourhood Σ . Take a C^2 -domain $\Omega' \subset \Omega$ such that $\partial\Omega' \cap \partial\Omega = \Sigma$ and define

$$J_\Sigma(\Omega') \stackrel{\text{def}}{=} \{\varphi \in J : \varphi|_{\Omega \setminus \overline{\Omega'}} \equiv 0\}. \quad (2.23)$$

Take $\varphi \in J_\Sigma(\Omega')$, multiply it by (2.21), integrate over Ω and apply Gauß' theorem. This leads to

$$\frac{\nu}{2} E(u, \varphi) = \langle f, \varphi \rangle_{L^2} + \int_{\partial\Omega} \varphi \cdot t \cdot n \, ds.$$

Because of $\varphi|_{\partial\Omega \setminus \Sigma} = 0$ the boundary integral is reduced to an integral over Σ and we obtain

$$\frac{\nu}{2} E(u, \varphi) = \langle f, \varphi \rangle_{L^2} + \int_{\Sigma} \varphi \cdot t \cdot n \, ds. \quad (2.24)$$

Since u is a weak solution, u satisfies (2.19). Comparing (2.19) with (2.24) we can conclude that

$$\int_{\Sigma} \varphi \cdot t \cdot n \, ds = \int_{\Sigma} ((\varphi \cdot \tau_1)\gamma_1 + (\varphi \cdot \tau_2)\gamma_2) \, ds, \quad \forall \varphi \in J_{\Sigma}(\Omega'). \quad (2.25)$$

Now we take a function $b \in (H^{1/2}(\Sigma))^3$ which is defined by

$$b_k \stackrel{\text{def}}{=} (\tau_k \cdot t \cdot n - \gamma_k)\tau_k, \quad (k = 1, 2).$$

Furthermore let us take a $\psi \in C^2(\partial\Omega')$ which is nontrivial and nonnegative with $\psi|_{\partial\Omega' \setminus \Sigma} \equiv 0$. Now we can extend b_k to all of $\partial\Omega'$ by zero outside Σ and we obtain that

$$a_k \stackrel{\text{def}}{=} b_k \psi \in (H^{1/2}(\partial\Omega'))^3 \quad \text{for } k = 1, 2.$$

Since the b_k are lying in the tangent plane of Γ and ψ is a scalar function, it follows that

$$\int_{\partial\Omega'} a_k \cdot n \, ds = 0, \quad k = 1, 2.$$

Therefore Lemma 2.32 is applicable to $a_k \in (H^{1/2}(\partial\Omega'))^3$ and we can conclude that

$$\exists \varphi_k \in (H^1(\Omega'))^3 \text{ with } \operatorname{div} \varphi_k = 0 \text{ and } \varphi_k|_{\partial\Omega'} = a_k.$$

We can extend φ_k to all of Ω by zero outside Ω' . Calling this extension again φ_k we have constructed a function $\varphi_k \in J_{\Sigma}(\Omega')$ with

$$\varphi_k|_{\partial\Omega \setminus \Sigma} \equiv 0 \text{ and } \varphi_k|_{\Sigma} = (\tau_k \cdot t \cdot n - \gamma_k)\psi\tau_k.$$

Substituting those particular φ_k into (2.25) we obtain

$$\int_{\Sigma} \varphi_k \cdot t \cdot n \, ds = \int_{\Sigma} (\tau_k \cdot t \cdot n - \gamma_k)\psi\tau_k \cdot t \cdot n \, ds = \int_{\Sigma} (\tau_k \cdot t \cdot n - \gamma_k)\psi\gamma_k \, ds.$$

Subtracting the right hand side of this equation from the left hand side leads to

$$\int_{\Sigma} (\tau_k \cdot t \cdot n - \gamma_k)^2 \psi \, ds = 0.$$

This implies that

$$\tau_k \cdot t \cdot n - \gamma_k = 0 \quad \text{a.e.,}$$

since ψ is nonnegative.

Periodicity of the pressure

To show the periodicity of p we shall follow the lines of the proof above with a different Ω' . Let $\Omega' \subset \Omega$ be a C^2 -domain with $\partial\Omega' \cap \partial\Omega = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1 \subset \{x = 0\} \cap \partial\Omega$ and $\Sigma_2 \subset \{x = x_0\} \cap \partial\Omega$. Moreover $\Sigma_2 = (x_0, 0, 0) + \Sigma_1$. Take any $\varphi \in J_{\Sigma_1 \cup \Sigma_2}(\Omega')$, multiply it by (2.21) and integrate over Ω . Then we obtain the equation corresponding to (2.25) which is

$$\int_{\Sigma_1 \cup \Sigma_2} \varphi \cdot t \cdot n \, ds = 0, \quad \forall \varphi \in J_{\Sigma_1 \cup \Sigma_2}(\Omega').$$

Since $\partial^\alpha u$ is periodic this integral reduces to

$$\int_{\Sigma_1 \cup \Sigma_2} \varphi \cdot (p n) \, ds = 0, \quad \forall \varphi \in J_{\Sigma_1 \cup \Sigma_2}(\Omega'). \quad (2.26)$$

Take a $\psi \in C^2(\Omega')$ with

- (i) $\psi|_{\partial\Omega' \setminus (\Sigma_1 \cup \Sigma_2)} \equiv 0$,
- (ii) ψ is nonnegative and nontrivial,
- (iii) $\psi|_{\Sigma_1} = \psi|_{\Sigma_2}$.

Furthermore define a function δ_p on $\Sigma_1 \cup \Sigma_2$ by the following:

If $X_1 = (0, y, z) \in \Sigma_1$ and $X_2 = (x_0, y, z) \in \Sigma_2$ are corresponding points, then:

$$\delta_p(X_1) = \delta_p(X_2) \stackrel{\text{def}}{=} p(X_1) - p(X_2).$$

We define now $a \in (H^{1/2}(\partial\Omega'))^3$ by $a|_{\Sigma_1} = \delta_p \psi n$ and $a|_{\Sigma_2} = -\delta_p \psi n$. Notice that $n|_{\Sigma_1} = -n|_{\Sigma_2}$. Therefore we have that $a|_{\Sigma_1} = a|_{\Sigma_2}$ by definition of a and

$$\int_{\partial\Omega'} a \cdot n \, ds = \int_{\Sigma_1 \cup \Sigma_2} a \cdot n \, ds = 0.$$

Hence we can apply Lemma 2.32 and obtain that

$$\exists \varphi \in (H^1(\Omega'))^3 \text{ with } \operatorname{div} \varphi = 0 \text{ and } \varphi|_{\partial\Omega'} = a.$$

As before we can extend φ to all of Ω by zero outside Ω' . So we have constructed a $\varphi \in J_{\Sigma_1 \cup \Sigma_2}(\Omega')$ which we now substitute into (2.26) and obtain

$$\begin{aligned} 0 &= \int_{\Sigma_1 \cup \Sigma_2} \varphi \cdot (p n) \, ds = \int_{\Sigma_1} \psi \delta_p p \, ds - \int_{\Sigma_2} \psi \delta_p p \, ds = \\ &= \int_{\Sigma_1} \psi \delta_p (p|_{\Sigma_1} - p|_{\Sigma_2}) \, ds = \int_{\Sigma_1} \psi (\delta_p)^2 \, ds. \end{aligned}$$

This implies that $\delta_p = 0$ a.e. on Σ_1 and Σ_2 or that $p|_{\Sigma_1} = p|_{\Sigma_2}$ a.e.. ■

Theorem 2.31 completes the weak formulation. In order to apply the weak formulation to the linearized problem (3.8) we need to show that

$$l(\varphi) \stackrel{\text{def}}{=} \langle f, \varphi \rangle_{L^2} + \int_{\Gamma} ((\varphi \cdot \tau_1)\gamma_1 + (\varphi \cdot \tau_2)\gamma_2) ds$$

defines a bounded linear functional on J . This will be shown in the next chapter. Here we move ahead assuming this result and state the weak formulation for the linearized problem.

Theorem 2.33

For Ω given and $f \in (L^2(\Omega))^3$ (i) and (ii) below are equivalent:

- (i) $\exists p \in L^2(\Omega)$ such that $(u, p) \in J \times L^2(\Omega)$ is a solution of (2.10) in the distributional sense. Moreover if $u \in (H^2(\Omega))^3$ and $p \in H^1(\Omega)$, then the boundary conditions (2.11) are satisfied as well.
- (ii) $u \in J$ is a weak solution of (2.10 and 2.11), i.e.:

$$\frac{\nu}{2} E(u, \varphi) = l(\varphi), \quad \forall \varphi \in J.$$

Chapter 3

Existence, uniqueness and regularity

In Theorem 2.31 we have obtained the result that u is a solution of the Stokes problem iff u is a weak solution, i.e. u satisfies

$$\frac{\nu}{2} E(u, \varphi) = \langle f, \varphi \rangle_{L^2} + \int_{\Gamma} ((\varphi \cdot \tau_1) \gamma_1 + (\varphi \cdot \tau_2) \gamma_2) ds, \quad \forall \varphi \in J. \quad (3.1)$$

In this chapter we shall show first that the right hand side of (3.1) is a linear functional on J . Having shown this result existence and uniqueness are an immediate consequence of Riesz' representation theorem. In the main part of this chapter we shall be concerned with the regularity of the weak solution u . There are already regularity results known for the Stokes problem. Since the Stokes problem satisfies the complementing boundary conditions of Agmon-Douglis-Nirenberg [Sol] we can apply the general regularity theory for elliptic systems [Agm/Dou/Nir]. Furthermore Solonnikov and Scadilov [Sol/Sca] have given a regularity proof which is not based on representation formulas. We shall give a regularity proof which is also not based on the results of [Agm/Dou/Nir]. Using the assumption that the free boundary Γ is globally the graph of the function $1+\zeta$, it turns out that the proof is fairly elementary even though there are a lot of technical details, which need to be worked out.

3.1 Existence and uniqueness

We shall show that the boundary integral of (3.1) is a bounded linear functional on J whenever γ_i are smooth enough.

Lemma 3.1.

Let $\varphi \in J$ and $\gamma : \Gamma_b \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a L^2 -function. Then the mapping $\varphi \mapsto \int_{\Gamma} (\varphi \cdot \tau) \gamma ds$ is a bounded linear functional on J . Here τ is a unit tangential vector on Γ .

Proof:

Since the embedding $(H^1(\Omega))^3 \hookrightarrow (L^2(\partial\Omega))^3$ is continuous, $\exists c_1 > 0$ such that

$$\|\varphi\|_{L^2(\partial\Omega)}^2 \leq c_1 \|\varphi\|_{H^1(\Omega)}^2 \leq c_2 \|\varphi\|_E^2, \quad \forall \varphi \in J.$$

The first inequality holds $\forall \varphi \in (H^1(\Omega))^3$ and the second $\forall \varphi \in J$ because of the equivalence of $\|\cdot\|_{H^1}$ and $\|\cdot\|_E$ in J . We can estimate the boundary integral as

$$\begin{aligned} \left| \int_{\Gamma} (\varphi \cdot \tau) \gamma ds \right| &\leq \int_{\Gamma} |\varphi \cdot \tau| |\gamma| ds \leq \left(\int_{\Gamma} |\varphi|^2 ds \right)^{1/2} \left(\int_{\Gamma} |\gamma|^2 ds \right)^{1/2} \\ &\leq \|\gamma\|_{L^2(\Gamma)} \|\varphi\|_{L^2(\partial\Omega)}^2 \leq c_2 \|\gamma\|_{L^2(\Gamma)} \|\varphi\|_E^2. \end{aligned}$$

Hence the right hand side of (3.1) is a bounded linear functional on J . Then Riesz' representation theorem is applicable to the bounded linear problem with homogeneous Dirichlet data at Γ_b . We shall show now that those boundary data ϕ_0 can be extended from Γ_b to a solenoidal vector field on Ω . Therefore it will be enough to solve the homogeneous linear problem.

Lemma 3.2

Let $\phi_0 \in (H^{1/2}(\Gamma_b))^3$ be a doubly periodic function with $\phi_0 \cdot n = 0$, where n is the outer unit normal on Γ_b . Then there exists an extension

$$\hat{\phi}_0 \in J \subset (H^1(\Omega))^3$$

of ϕ_0 , i.e.,

$$\hat{\phi}_0|_{\Gamma_b} = \phi_0.$$

Proof:

First we extend ϕ_0 to $\partial\Omega_{1/2}$ in such a way that

$$\int_{\partial\Omega_{1/2}} \phi_0 \cdot n ds = 0, \quad \text{where} \quad \Omega_{1/2} = \{(x, y, z) \in \Omega : |z| \leq 1/2\}.$$

If $\Omega_{1/2}$ would be a smooth domain, let us say a C^2 -domain, then Lemma 2.32 could be applied. In our case $\Omega_{1/2}$ is only a $C^{0,1}$ -domain. We shall show that an extension $\hat{\phi}_0$ can be constructed in the same way as in the proof of Lemma 2.32 (see [Lad, p.24 ff.]). From

$$\phi_0 \cdot n|_{\Gamma_b} = 0$$

we conclude that $\phi_0 = (\phi_{01}, \phi_{02}, 0)$. Furthermore ϕ_{0i} , ($i = 1, 2$), are doubly periodic by assumption. We multiply ϕ_0 with a bump function $\chi(z) \in C^\infty([0, 1])$ which satisfies

$$\chi(z) = 1, \quad \text{for } 0 \leq z \leq 1/8,$$

$$\chi(z) = 0, \quad \text{for } 1/4 \leq z.$$

The vectorfield $\chi(z)\phi_0(x, y)$ is now naturally defined on

$$\Gamma_b \cup \Gamma_x \cup \Gamma_y,$$

where

$$\Gamma_x = (\{x = 0\} \cup \{x = x_0\}) \cap \partial\Omega,$$

$$\Gamma_y = (\{y = 0\} \cup \{y = 1\}) \cap \partial\Omega.$$

Moreover, we may extend $\chi\phi_0$ to all of $\partial\Omega_{1/2}$ by zero on $\{z = 1/2\} \cap \Omega$. Then we obtain that

$$\int_{\partial\Omega_{1/2}} \chi\phi_0 \cdot n \, ds = 0.$$

Following the proof, given in [Lad, p.24 ff.], we decompose $\chi\phi_0$ into a normal and a tangential vectorfield with respect to $\partial\Omega_{1/2}$, i.e.,

$$\chi\phi_0 = \phi_n + \phi_\tau,$$

where ϕ_n is the normal vector field and ϕ_τ the tangential vector field. We define

$$\varphi_n \stackrel{\text{def}}{=} \phi_n \cdot n$$

and consider the Neumann problem

$$\begin{aligned} \Delta\psi &= 0, \quad \text{in } \Omega_{1/2} \\ \frac{\partial\psi}{\partial n} &= \varphi_n, \quad \text{on } \partial\Omega_{1/2}. \end{aligned} \tag{3.2}$$

By the regularity theory we can conclude that $\psi \in H_{loc}^2(\Omega_{1/2})$. Using Corollary 2.13 we obtain that $\psi \in H^1(\Omega_{1/2})$. Now we glue copies $\Omega_{1/2,i}$, ($i = 1, \dots, 14$), of $\Omega_{1/2}$ around $\Omega_{1/2}$ in such a way that $\Omega_{1/2}$ is lying completely in the interior of the new domain

$$\Omega_e = \Omega_{1/2} \bigcup \left(\bigcup_{i=1}^{14} \Omega_{1/2,i} \right).$$

Each copy is symmetric with respect to each neighbour through the common side. By reflecting the partial derivatives $\frac{\partial \psi}{\partial x_i}$ through each common side starting from $\Omega_{1/2}$ we can extend all the partial derivatives to Ω_e . Hence by integrating these derivatives we have extended ψ to $\psi_e \in H^1(\Omega_e)$. Now we take a C^2 - subdomain $\tilde{\Omega}$ with

$$\Omega_{1/2} \subset \subset \tilde{\Omega} \subset \subset \Omega_e$$

and consider the Dirichlet problem

$$\begin{aligned} -\Delta \xi &= 0 & \text{in } \tilde{\Omega} \\ \xi &= \psi_e|_{\partial \tilde{\Omega}}. \end{aligned} \tag{3.3}$$

The fact that

$$\psi \in H_{loc}^2(\Omega_{1/2}),$$

implies that

$$\psi_e|_{\partial \tilde{\Omega}} \in H^1(\partial \tilde{\Omega}).$$

Therefore the solution ξ of (3.3) is an element of

$$H^1(\tilde{\Omega}) \cap H_{loc}^2(\tilde{\Omega}).$$

Moreover we have by construction that

$$\frac{\partial \xi}{\partial n} \Big|_{\partial \Omega_{1/2}} = \frac{\partial \psi}{\partial n} \Big|_{\partial \Omega_{1/2}}.$$

Since the solution of the Neumann problem is unique up to a constant and $\xi|_{\Omega_{1/2}}$ and ψ are solutions of (3.2) we can assume w.l.o.g. that

$$\xi|_{\Omega_{1/2}} = \psi.$$

This implies that

$$\psi \in H^2(\Omega_{1/2}).$$

in a similar way it can be shown that there exists a vectorfield $d \in (H^2(\Omega_{1/2}))^3$ such that

$$\operatorname{curl} d = \phi_\tau \quad \text{on } \partial\Omega_{1/2}.$$

Hence we have obtained an extension

$$\hat{\phi}_0 = \operatorname{grad} \psi + \operatorname{curl} d, \quad (3.4)$$

$$\hat{\phi}_0 = \phi_0 \quad \text{on } \partial\Omega_{1/2} \quad (3.5)$$

$$\operatorname{div} \hat{\phi}_0 = 0. \quad (3.6)$$

Furthermore we have that $\hat{\phi}_0|_{\{z=1/2\}} \equiv 0$, which implies that

$$\frac{\partial \hat{\phi}_0}{\partial x} = \frac{\partial \hat{\phi}_0}{\partial y} = 0 \quad \text{on } \{z = 1/2\}.$$

if the traces are defined. Because of (3.6) also the third partial derivative is zero at $\{z = 1/2\}$. Hence we can extend $\hat{\phi}_0$ to all of Ω by zero outside $\Omega_{1/2}$ which proves the assertion.

We remark that also the regularity in the Hölder spaces could be treated in the same way, which leads for example to

$$\hat{\phi}_0 \in (C^{4,\alpha}(\Omega))^3, \text{ if } \phi_0 \in (C^{3,\alpha}(\Gamma_b))^3.$$

■

Now we may apply Riesz' representation theorem and identify elements of $(H^{-1}(\Omega))^3$ with elements of $(H^1(\Omega))^3$.

Theorem 3.3

Let $f \in (H^{-1}(\Omega))^3$, $\zeta \in C^{3,\alpha}(\Gamma_b)$ (see (1.12), γ a continuous function with respect to ζ and ϕ_0 be the extension of the boundary data ϕ_0 constructed in Lemma 3.2. Then there exists a unique solution $u' \in J$ for the homogeneous Stokes problem (3.7). Moreover, $u = u' + \hat{\phi}_0$ is a solution of (2.12).

Proof:

We first construct the extension $\hat{\phi}_0$ of the inhomogeneous Dirichlet data ϕ_0 as in Lemma 3.2. Then we define

$$f_s \stackrel{\text{def}}{=} f + \Delta \hat{\phi}_0$$

and consider the homogeneous Stokes problem

$$\left. \begin{aligned} -\Delta u' + \operatorname{grad} p &= f_s \\ \operatorname{div} u' &= 0 \\ u'|_{\Gamma_b} &= 0 \end{aligned} \right\} \text{ in } \Omega \quad \left. \vphantom{\begin{aligned} -\Delta u' + \operatorname{grad} p &= f_s \\ \operatorname{div} u' &= 0 \\ u'|_{\Gamma_b} &= 0 \end{aligned}} \right\} \quad (3.7)$$

and the remaining boundary conditions of (2.11).

By Theorem 2.31 u' is a solution of (3.7) iff u' is a weak solution of (3.7). Since the right hand side of the integral equation in the weak formulation

$$\langle f_s, \varphi \rangle_{L^2} + \int_{\Gamma} (\varphi \cdot \tau) \gamma \, ds$$

defines a bounded linear functional on J , $\exists!$ weak solution $u' \in J$ of (3.7) by Riesz' representation theorem (see above). The extension ϕ_0 has been constructed in a way that it is now obvious that $u = u' + \phi_0$ is a solution of (2.12). ■

Before we proof the next existence result we want to switch to the dimensionless formulation of our problem. Therefore we need to define reference values and parameters.

Definition 3.4

We define the reference values for the length, speed and pressure as:

$$d_R = 1 = \text{height of the fluid film,}$$

$$u_R = u_0|_{\Gamma_0, c=0} = \frac{ad_R^2}{2\nu},$$

$$p_R = u_R^2.$$

The parameters are defined as:

$$\lambda = \frac{u_R d_R}{\nu} = \frac{ad_R^3}{2\nu}, \quad (\text{Reynolds number}),$$

$$\mu_i = \frac{c_i d_R}{\nu}, \quad (i = 1, 2),$$

$$\theta = \frac{gd_R^3}{\nu^2},$$

$$\sigma = \frac{\kappa d_R}{\nu^2}.$$

The linearized problem in dimensionless form is given by:

$$\left. \begin{aligned} -\frac{1}{\lambda} \Delta u + \text{grad } p + u_{0k} \partial_k u + u_k \partial_k u_0 &= f \\ \text{div } u &= 0 \end{aligned} \right\} \quad \text{in } \Omega \quad (3.8)$$

$$\left. \begin{aligned} \left. \begin{aligned} \partial^\alpha u|_{x=0} &= \partial^\alpha u|_{x=x_0} \\ \partial^\alpha u|_{y=0} &= \partial^\alpha u|_{y=1} \end{aligned} \right\} |\alpha| \leq 1 \quad (B1) \\ p|_{x=0} &= p|_{x=x_0} \\ p|_{y=0} &= p|_{y=1} \\ u|_{\Gamma_b} &= \phi_0 \quad (B2) \\ u \cdot n|_{\Gamma} &= 0 \quad (B3) \\ \tau_k \cdot t(u) \cdot n|_{\Gamma} &= \gamma_k, \quad k = 1, 2 \end{aligned} \right\} \quad (3.9)$$

with

$$u_0 = \begin{pmatrix} \frac{2z}{(1+\zeta)^2} - \frac{z^2}{(1+\zeta)^3} - \frac{\mu_1}{\lambda} \frac{1}{1+\zeta} \\ -\frac{\mu_2}{\lambda} \frac{1}{1+\zeta} \\ \frac{2z^2\zeta_x}{(1+\zeta)^3} - \frac{z^3\zeta_x}{(1+\zeta)^4} - \frac{\mu_1}{\lambda} \frac{z\zeta_x}{(1+\zeta)^2} - \frac{\mu_2}{\lambda} \frac{z\zeta_y}{(1+\zeta)^2} \end{pmatrix}, \quad (3.10)$$

$$p_0 = -\frac{2}{\lambda} + \frac{\theta}{\lambda^2} \left(1 - \frac{z}{1+\zeta} \right), \quad (3.11)$$

$$\phi_0 = \begin{pmatrix} \frac{\mu_1}{\lambda} \frac{\zeta}{1+\zeta} \\ \frac{\mu_2}{\lambda} \frac{\zeta}{1+\zeta} \\ 0 \end{pmatrix}, \quad (3.12)$$

$$\gamma_1 = -2 \frac{(1 - \frac{\mu_1}{\lambda}) \zeta_{xx} - \frac{\mu_2}{\lambda} \zeta_{xy}}{\sqrt{1+\zeta} \sqrt{1+\zeta_x^2 + \zeta_y^2}} \quad (3.13)$$

$$\gamma_2 = -2 \frac{(1 - \frac{\mu_1}{\lambda}) [(1+\zeta_x^2)\zeta_{xy} - \zeta_x\zeta_y\zeta_{xy}] - \frac{\mu_2}{\lambda} [(1+\zeta_x^2)\zeta_{yy} - \zeta_x\zeta_y\zeta_{yy}]}{(1+\zeta)(1+\zeta_x^2 + \zeta_y^2)\sqrt{1+\zeta_x}} \quad (3.14)$$

The homogeneous Stokes problem becomes:

$$\left. \begin{aligned} -\frac{1}{\lambda} \Delta u + \operatorname{grad} p &= f \\ \operatorname{div} u &= 0 \end{aligned} \right\} \quad \text{in } \Omega, \quad (3.15)$$

$$\begin{aligned} u|_{\Gamma_b} &= 0 \\ u &\in V. \end{aligned} \quad (3.16)$$

Let us define the linear first order differential operator L as

$$Lu \stackrel{\text{def}}{=} \sum_{k=1}^3 (u_{0k} \partial_k u + u_k \partial_k u_0).$$

We decompose this operator into a sum of three operators:

$$L_0 u \stackrel{\text{def}}{=} \sum_{k=1}^3 (u_{0k}^0 \partial_k u + u_k \partial_k u_0^0), \quad (3.17)$$

$$\frac{\mu_i}{\lambda} L_i \stackrel{\text{def}}{=} \sum_{k=1}^3 (u_{0k}^i \partial_k u + u_k \partial_k u_0^i), \quad (i = 1, 2), \quad (3.18)$$

where

$$u_0^1 = - \begin{pmatrix} \frac{1}{1+\zeta} \\ 0 \\ \frac{z\zeta_x}{(1+\zeta)^2} \end{pmatrix}, \quad u_0^2 = - \begin{pmatrix} 0 \\ \frac{1}{1+\zeta} \\ \frac{z\zeta_y}{(1+\zeta)^2} \end{pmatrix}$$

and

$$u_0^0 = \begin{pmatrix} \frac{2z}{(1+\zeta)^2} - \frac{z^2}{(1+\zeta)^3} \\ 0 \\ \frac{2z^2\zeta_x}{(1+\zeta)^3} - \frac{z^3\zeta_x}{(1+\zeta)^4} \end{pmatrix}.$$

Then the weak formulation of the linear problem is:

$$\begin{aligned} & \frac{1}{\lambda} \left\{ \frac{1}{2} E(u', \varphi) + \mu_1 \langle L_1 u', \varphi \rangle_{L^2} + \mu_2 \langle L_2 u', \varphi \rangle_{L^2} \right\} \\ &= \langle f, \varphi \rangle_{L^2} - \langle L_0 u', \varphi \rangle_{L^2} + \langle \gamma, \varphi \rangle_{L^2(\Gamma)}, \quad \forall \varphi \in J. \\ & u = u' + \phi_0, \end{aligned} \quad (3.19)$$

with ϕ_0 a suitable extension of $\phi_0|_{\Gamma_b}$ (see Lemma 3.2) and

$$f = \tilde{f} - (-\Delta\phi_0 + L\phi_0),$$

where $\tilde{f}$ is f in (3.8). For convenience we set $\lambda = 1$. In order to show the existence of a solution of (3.19) we anticipate the regularity result for the solution of the Stokes problem (3.7). The regularity result will be given in the next section. By Theorem 3.3 there exists a solution operator

$$K_S : (L^2(\Omega))^3 \rightarrow J, \quad f \mapsto K_S f = u'$$

for the Stokes problem (3.7). Now we apply the regularity result for u' (see Theorem 3.10 below) and obtain that $u' \in J \cap (H^2(\Omega))^3$. Hence the operator

$$K_S : (L^2(\Omega))^3 \rightarrow J \subset (H^1(\Omega))^3$$

is a compact operator. So we are able to apply the Riesz - Schauder theory to the linear problem. We want to consider (3.19) without the term $\langle L_0 u', \varphi \rangle_{L^2}$.

Theorem 3.5

Let $f \in (L^2(\Omega))^3, \gamma \in (H^1(\Gamma))^3$ and $K_S : (L^2(\Omega))^3 \rightarrow J$ the solution operator of the Stokes problem (3.7). Then there exists a unique solution $u \in J$ of

$$\begin{aligned} \frac{1}{2}E(u, \varphi) + \mu_1 \langle L_1 u, \varphi \rangle_{L^2} + \mu_2 \langle L_2 u, \varphi \rangle_{L^2} &= \langle f, \varphi \rangle_{L^2} + \langle \gamma, \varphi \rangle_{L^2(\Gamma)}, \\ \forall \varphi \in J, \end{aligned} \quad (3.20)$$

whenever the following two conditions are satisfied:

- (i) μ_1 is not an eigenvalue of $K_S \circ L_1 : J \rightarrow J$,
- (ii) μ_2 is not an eigenvalue of $(I + \mu_1 K_S \circ L_1)^{-1} K_S L_2 : J \rightarrow J$.

Proof:

We apply the operator K_S to (3.20) and obtain the operator equation

$$(I + \mu_1 K_S \circ L_1 + \mu_2 K_S \circ L_2)u = K_S f.$$

If μ_1 is not an eigenvalue of $K_S \circ L_1 : J \rightarrow J$, then $I + \mu_1 K_S \circ L_1$ can be inverted by the Riesz-Schauder theory. Notice that $K_S \circ L_1$ is a compact operator since K_S is compact. Hence the operator equation above becomes

$$(I + \mu_2(I + \mu_1 K_S \circ L_1)^{-1} K_S \circ L_2)u = (I + \mu_1 K_S \circ L_1)^{-1} K_S f.$$

If (ii) is satisfied, then

$$I + \mu_2(I + \mu_1 K_S \circ L_1)^{-1} K_S \circ L_2$$

can be inverted by the same arguments as above. ■

3.2 Regularity of the solution of the Stokes problem

In this section we are concerned with the regularity of the solution of the homogeneous Stokes problem:

$$\left. \begin{aligned} -\Delta u + \text{grad } p &= f \\ \text{div } u &= 0 \end{aligned} \right\} \text{ in } \Omega$$

periodic boundary conditions as in (3.9)

$$\left. \begin{aligned} u|_{\Gamma_b} &= 0 \\ u \cdot n|_{\Gamma} &= 0 \\ \tau_k \cdot t(u) \cdot n|_{\Gamma} &= \gamma_k. \end{aligned} \right\} \quad (3.21)$$

Definition 3.6

Let $a(u, v)$ be a continuous bilinear form on $(H^1(\Omega))^3 \times (H^1(\Omega))^3$. Then $a(u, v)$ is called J - coercive iff

$\exists$ constants $c_1, c_2 > 0$ and $k \geq 0$ such that

(i) $|a(u, v)| \leq c_1 \|u\|_{H^1} \|v\|_{H^1}$

and

(ii) $a(u, v) + k\langle u, u \rangle_{L^2} \geq c_2 \|u\|_{H^1}^2, \quad \forall u \in J.$

Considering the boundary conditions which are contained in J , we observe that the periodic conditions at the sides $\{x = 0\}$, $\{x = x_0\}$, $\{y = 0\}$ and $\{y = 1\}$ as well as the zero Dirichlet boundary condition at Γ_b allow us to obtain estimates up to the boundary by interior estimates. Therefore the only boundary which might cause some trouble is Γ . At Γ the condition $v \cdot n = 0$ must be satisfied. Let us now outline the scheme for the regularity proof:

- The boundary value problem (BVP) is transformed onto the domain Ω_0 . This can be done by a transformation which maps solenoidal vector fields into solenoidal ones (see the transformation of $U_0 \mapsto u_0$ (1.42) below).

- A weak formulation for the transformed BVP on Ω_0 is given.
- By using a cut-off function the integral equation is transformed into an equation on Q , where Q is a cubic neighbourhood of x_0 and x_0 is an arbitrary point in the open set $\overset{\circ}{\Gamma}_0$.
- Estimates for the derivatives of $\partial_i u$ in tangential direction with respect to Γ_0 are given, i.e., in x_1 - and x_2 - directions.

The estimates for the normal derivatives are then obtained from the differential equations. The scheme for the proof is very much like the standard procedure one could expect and find e.g. in [Wlo, p. 298].

3.2.1 Transformation onto Ω_0

Let $\Phi : \Omega_0 \rightarrow \Omega$ and $\Psi : \Omega \rightarrow \Omega_0$ be the transformation mappings given by (1.12) and (1.19), respectively. A vector field v on Ω will be transformed as a contravariant vector, i.e.,

$$v^i \mapsto u^i = \frac{\partial \Psi^i}{\partial y^j} (\Phi(x)) \cdot v^j (\Phi(x)). \quad (3.22)$$

We remark that $\operatorname{div} v = 0$ does not imply that $\operatorname{div} u = 0$. From the discussion about the transformation of the basic solution (see (1.42)) we know that $\sqrt{g} u$ is a solenoidal vector field, where g is the determinant of the metric tensor. When we transform v into u , the BVP on Ω is transformed into the following BVP on Ω_0 :

$$\left. \begin{aligned} & - \left(g^{kl} u^i_{,l} \right)_{,k} + g^{ik} p_{,k} = f^i \\ & u^i_{,i} = 0 \\ & \text{periodic boundary conditions as in (3.9)} \\ & u|_{\Gamma_b} = 0 \\ & u^i n_i|_{\Gamma_0} = 0 \\ & \tau_k t^{ik}(u) n_i|_{\Gamma_0} = \gamma_k, \quad (k = 1, 2). \end{aligned} \right\} \quad \text{in } \Omega_0 \quad (3.23)$$

3.2.2 The weak formulation on Ω_0

Let $\varphi \in J(\Omega_0)$. For obtaining the weak formulation on Ω we have used the identity (see (2.14) ff.)

$$\nabla(t \cdot \varphi) = (\operatorname{div} t) \cdot \varphi + t \cdot \operatorname{grad} \varphi.$$

In Ω_0 this identity becomes

$$\left(t^{ik} \varphi_i\right)_{,k} = t^{ik}_{,k} \varphi_i + t^{ik} \varphi_{i,k}.$$

Hence $-\operatorname{div} t = -\Delta u + \operatorname{grad} p$ turns into

$$-t^{ik}_{,k} = -\left(g^{kl} u^i_{,l}\right)_{,k} + g^{il} p_{,l}. \quad (3.24)$$

Furthermore we might apply $\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} = \Gamma^s_{sk}$ (see (1.26)) and

$$\left(t^{ik} \varphi_i\right)_{,k} = \frac{\partial \left(t^{ik} \varphi_i\right)}{\partial x^k} + \Gamma^s_{sk} t^{is} \varphi_i.$$

Then we can write:

$$\sqrt{g} \left(t^{ik} \varphi_i\right)_{,k} = \operatorname{div} \left(\sqrt{g} t^{ik} \varphi_i\right). \quad (3.25)$$

Using those equations above we obtain:

$$\begin{aligned} \int_{\partial\Omega_0} \sqrt{g} t^{ik} \varphi_i n_k ds &= \int_{\Omega_0} \operatorname{div} \left(\sqrt{g} t^{ik} \varphi_i\right) dx = \int_{\Omega_0} \sqrt{g} \left(t^{ik} \varphi_i\right)_{,k} dx \\ &= \int_{\Omega_0} \sqrt{g} t^{ik}_{,k} \varphi_i dx + \int_{\Omega_0} \sqrt{g} t^{ik} \varphi_{i,k} dx \\ &= - \int_{\Omega_0} \sqrt{g} \left\{ -\left(g^{kl} u^i_{,l}\right)_{,k} + g^{il} p_{,l} \right\} \varphi_i dx + \int_{\Omega_0} \sqrt{g} t^{ik} \varphi_{i,k} dx. \end{aligned} \quad (3.26)$$

By (3.23) the first integral on the right hand side of (3.26) is

$$- \int_{\Omega_0} \sqrt{g} f^i \varphi_i dx.$$

In order to compute the second integral on the right hand side of (3.26) we compute first $\varphi_{i,k}$ and $\sqrt{g} t^{ik}$ separately:

$$\begin{aligned} \varphi_{i,k} &= \frac{\partial \varphi_i}{\partial x^k} - \Gamma^r_{ik} \varphi_r, \\ t^{ik}(u) &= -p g^{ik} + \left(g^{kl} u^i_{,l} + g^{il} u^k_{,l}\right) \\ &= -p g^{ik} + \left(g^{kl} \frac{\partial u^i}{\partial x^l} + g^{il} \frac{\partial u^k}{\partial x^l}\right) + \left(g^{kl} \Gamma^i_{ls} u^s + g^{il} \Gamma^k_{ls} u^s\right), \end{aligned}$$

and

$$\frac{\partial u^i}{\partial x^l} = \frac{1}{\sqrt{g}} \frac{\partial v^i}{\partial x^l} - \frac{1}{g} \frac{\partial \sqrt{g}}{\partial x^l} v^i,$$

where

$$v = \sqrt{g} u \in J(\Omega_0).$$

This implies that

$$\left. \begin{aligned} \int_{\Omega_0} \sqrt{g} t^{ik} \varphi_{i,k} dx &= - \int_{\Omega_0} \sqrt{g} p \underbrace{g^{ik} \varphi_{i,k}}_{\varphi^k_{,k}} dx + \\ &\int_{\Omega_0} \left(\frac{\partial v^i}{\partial x^l} g^{kl} \varphi_{i,k} + \frac{\partial v^k}{\partial x^l} g^{il} \varphi_{i,k} \right) dx + \\ &\int_{\Omega_0} \left(g^{kl} \Gamma_{ls}^i + g^{il} \Gamma_{ls}^k \right) v^s \frac{\partial \varphi_i}{\partial x^k} dx - \\ &\int_{\Omega_0} \left(g^{kl} \Gamma_{ls}^i + g^{il} \Gamma_{ls}^k \right) \Gamma_{ik}^r v^s \varphi_r - \\ &\int_{\Omega_0} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^l} \left(g^{kl} v^i + g^{il} v^k \right) \frac{\partial \varphi_i}{\partial x^k} dx + \\ &\int_{\Omega_0} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^l} \left(g^{kl} v^i + g^{il} v^k \right) \Gamma_{ik}^r \varphi_r dx. \end{aligned} \right\} \quad (3.27)$$

When we use Gauß' theorem and carry out the covariant derivatives, (3.27) becomes

$$\left. \begin{aligned} \int_{\Omega_0} \sqrt{g} t^{ik} \varphi_{i,k} dx &= \int_{\Omega_0} \left(\frac{\partial v^i}{\partial x^l} g^{kl} \frac{\partial \varphi_i}{\partial x^k} + \frac{\partial v^k}{\partial x^l} g^{il} \frac{\partial \varphi_i}{\partial x^k} \right) dx + \\ &\int_{\Omega_0} \frac{\partial \sqrt{g}}{\partial x^i} \varphi^i p dx + \\ &\int_{\Omega_0} (Lv)_i \varphi^i dx + \int_{\partial \Omega_0} (Mv)_i \varphi^i dx ds, \end{aligned} \right\} \quad (3.28)$$

where

- (i) $M : J(\Omega_0) \rightarrow J(\Omega_0)$ is a linear continuous operator which involves only multiplications with continuous functions and summations of such terms,
- (ii) $L : J(\Omega_0) \rightarrow L^2(\Omega)$ is a linear continuous operator which involves

also first order derivatives of v .

Let us call the first integral $E_\Phi(v, \varphi)$, i.e.:

$$E_\Phi(v, \varphi) = 2 \int_{\Omega_0} \left(g^{kl} \frac{\partial v^i}{\partial x^l} + g^{il} \frac{\partial v^k}{\partial x^l} \right) \frac{\partial \varphi_i}{\partial x^k} dx. \quad (3.29)$$

We claim that $E_\Phi(\cdot, \cdot)$ defines an inner product on $J(\Omega_0)$ in the same way $E(\cdot, \cdot)$ does it on $J(\Omega)$. First we observe that we may interchange i and k without changing E_Φ , which leads to

$$\left. \begin{aligned} E_\Phi(v, \varphi) &= \int_{\Omega_0} \left(g^{kl} \frac{\partial v^i}{\partial x^l} + g^{il} \frac{\partial v^k}{\partial x^l} \right) \left(\frac{\partial \varphi_i}{\partial x^k} + \frac{\partial \varphi_k}{\partial x^i} \right) dx \\ &= 2 \int_{\Omega_0} g^{kl} \frac{\partial v^i}{\partial x^l} \frac{\partial \varphi_i}{\partial x^k} dx + 2 \int_{\Omega_0} \frac{\partial v^i}{\partial x^j} \frac{\partial \varphi_j}{\partial x^i} dx. \end{aligned} \right\} \quad (3.30)$$

When we take $\varphi = v$, we observe that the second integral is exactly the one we have discussed in the proof of Lemma 2.25. In this proof we have shown that $E(\cdot, \cdot)$ defines an inner product on $J(\Omega)$. Hence we know how to estimate the second integral. The integrand in the first integral can be written as

$$g^{kl} \frac{\partial v^i}{\partial x^l} \frac{\partial v_i}{\partial x^k} = |\nabla v|_G^2,$$

where $|\cdot|_G$ is the norm on Ω_0 induced by the metric $G = (g^{ij})$. Since the Euclidean metric is equivalent to the G - metric, we conclude that there exists a positive function $c_0(x, y)$ such that

$$g^{kl} \frac{\partial v^i}{\partial x^l} \frac{\partial v_i}{\partial x^k} \geq c_0 |\nabla v|^2 = c_0 \sum_{i,j} \left(\frac{\partial v^i}{\partial x^j} \right)^2.$$

From the g^{kl} given in (1.15) we observe that c_0 is bounded above and bounded away from zero as long as ζ is strictly less than 1. Therefore $E_\Phi(\cdot, \cdot)$ defines an inner product on $J(\Omega_0)$ for small ζ . We write (3.27) in the form

$$\int_{\Omega_0} \sqrt{g} t^{ik} \varphi_{i,k} dx = E_\Phi(v, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} + \langle Lv, \varphi \rangle_{L^2} + \langle Mv, \varphi \rangle_{L^2(\partial\Omega)}.$$

Hence we may write (3.26) as the following integral equation:

$$\left. \begin{aligned} \int_{\Omega_0} \operatorname{div} (\sqrt{g} t^{ik} \varphi_i) dx = \\ \langle \hat{f}, \varphi \rangle_{L^2} + E_\Phi(v, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} + \langle Lv, \varphi \rangle_{L^2} + \langle Mv, \varphi \rangle_{L^2(\partial\Omega)}, \end{aligned} \right\} \quad (3.31)$$

where $\hat{f}_i = \sqrt{g} g_{ik} f^k$.

The left hand side of (3.31) is by Gauß' theorem equal to

$$\begin{aligned} \int_{\partial\Omega_0} \sqrt{g} t^{ik}(u) \varphi_i n_k ds &= \int_{\Gamma} \sqrt{g} t^{ik}(u) \varphi_i n_k ds = \\ \int_{\Gamma} \sqrt{g} (\tau_\alpha)_i t^{ik}(u) n_k (\tau^\alpha)^i \varphi_l ds &= \int_{\Gamma} \sqrt{g} \gamma_\alpha (\tau^\alpha)_i \varphi^i ds = \\ \int_{\Gamma} \sqrt{g} \gamma^i \varphi_i ds &= \int_{\Gamma} \sqrt{g} g_{ik} \gamma^i \varphi^k ds = \langle \hat{\gamma}, \varphi \rangle_{L^2(\Gamma)}, \end{aligned}$$

where $\hat{\gamma}_k = \sqrt{g} g_{ik} \gamma^i$.

Here we have used the tangential stress boundary conditions on Γ (see 3.21).

Now we can write (3.31) in its final form:

$$\left. \begin{aligned} E_\Phi(v, \varphi) + \langle Mv, \varphi \rangle_{L^2(\partial\Omega_0)} + \langle p\nabla g, \varphi \rangle_{L^2} \\ = \langle \hat{f} - Lv, \varphi \rangle_{L^2} + \langle \hat{\gamma}, \varphi \rangle_{L^2(\Gamma)}. \end{aligned} \right\} \quad (3.32)$$

3.2.3 Localization of the integral equation

Let $x_0 \in \overset{\circ}{\Gamma}_0$ be an interior point and χ be a cutoff function which is only nonzero in a neighbourhood of x_0 . Later on we shall need the following result which can be found for example in [Wlo, Satz 19.1].

Lemma 3.7

The sesquilinear form

$$\mathcal{A}(v, \varphi) \stackrel{\text{def}}{=} E_\Phi(v, \varphi) + \langle Mv, \varphi \rangle_{L^2(\partial\Omega_0)}$$

is a J -coercive form.

Remark 3.8

Define for $v \in J \subset (H^1(\Omega_0))^3$: $h \stackrel{\text{def}}{=} \hat{f} - Lv$. Then $h \in (L^2(\Omega_0))^3$, since L is a first order differential operator with continuous coefficients.

For convenience we shall call $\hat{\gamma}$ simply γ . Let us assume that $v \in J(\Omega_0)$ is a solution of

$$\mathcal{A}(v, \varphi) + \langle p\nabla g, \varphi \rangle_{L^2} = \langle h, \varphi \rangle_{L^2} + \langle \gamma, \varphi \rangle. \quad (3.33)$$

The proof that $v \in (H^2(\Omega_0))^3$ is nonstandard with respect to two points:

- (i) v is a solenoidal vector field,
- (ii) $\langle \gamma, \varphi \rangle_{L^2(\Gamma_0)}$ is a boundary integral.

Let $Q \subset \mathbb{R}^3$ be an open cubic neighbourhood of x_0 such that $Q \cap \partial\Omega_0 \subset \overset{\circ}{\Gamma}_0$. Furthermore assume that Q is a cube with half edge length ρ and x_0 is the center point of Q . We define $Q_{n\epsilon}$ to be the cube with center x_0 and half edge length $\rho - n\epsilon$ for $n = 1, 2$ and 3 . Take a bump function $\chi \in \mathcal{D}(Q_{2\epsilon})$ with $\chi \equiv 1$ in $Q_{3\epsilon}$ (see Figure 3.1). In order to estimate the tangential derivatives we shall estimate the tangential difference quotient

$$\Delta_h^{\tau_i} v = \frac{v(\dots, x_i + h, \dots) - v(\dots, x_i, \dots)}{h}$$

with $|h| \leq \epsilon$. The index τ_i means that we consider the difference quotient with respect to the i^{th} tangential direction of Γ_0 . Since it does not matter which tangential direction we consider we simply write Δ_h^{τ} . We can not substitute $\Delta_h^{\tau}(\chi u)$ into (3.33) for φ , because $\Delta_h^{\tau}(\chi u)$ might not be an element of $J(\Omega_0)$. Therefore we have to construct some corrections terms.

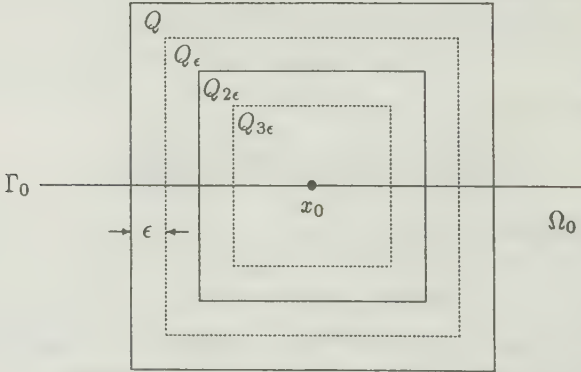


Figure 3.1: $Q_{3\epsilon} \subset Q_{2\epsilon} \subset Q_{\epsilon} \subset Q \subset \Gamma_0$, cubic neighbourhoods of x_0

Lemma 3.9

Let $v \in V$ and $\chi \in \mathcal{D}(Q_{2\epsilon})$.

Then there exist linear bounded operators

$$K_h^\tau, L_h^\tau : V \rightarrow (H^1(\Omega_0))^3$$

such that

- (i) $\Delta_h^\tau(\chi v) - K_h^\tau v \in J(\Omega_0)$ with support in Q_ϵ ,
- (ii) $\chi(\Delta_h^\tau v) - L_h^\tau v \in J(\Omega_0)$ with support in Q_ϵ ,
- (iii) $\exists$ a constant $c > 0$ such that $\|K_h^\tau\|, \|L_h^\tau\| \leq c < \infty$. The constant c is independent of h .

Proof:

If $\Delta_h^\tau(\chi v) - K_h^\tau v \in J(\Omega_0)$, then

$$\operatorname{div}(\Delta_h^\tau(\chi v) - K_h^\tau v) = 0$$

or

$$\operatorname{div}(K_h^\tau v) = \operatorname{div}(\Delta_h^\tau(\chi v)).$$

If we furthermore assume that $K_h^\tau v$ is the gradient of an H^2 -function w_h^τ , then the divergence condition above becomes:

$$\nabla^2 w_h^\tau \doteq \operatorname{div}(\Delta_h^\tau(\chi v)) \stackrel{\text{def}}{=} f_h^\tau. \quad (3.34)$$

The boundary condition on $\Gamma_0 - v \cdot n|_{\Gamma_0} = 0$ - is satisfied by χv and therefore also by $\Delta_h^\tau(\chi v)$ as one can see from the definition of Δ_h^τ . This implies that $K_h^\tau v$ must satisfy the same boundary condition. Therefore w_h^τ satisfies zero Neumann boundary condition on Γ_0 . Hence we can take for w_h^τ a solution of the mixed Dirichlet/Neumann problem

$$\left. \begin{aligned} \Delta w_h^\tau &= f_h^\tau && \text{in } Q_{2\epsilon}^- \\ \frac{\partial w_h^\tau}{\partial n} &= 0 && \text{on } \Gamma_0 \cap \partial Q_{2\epsilon}^- \\ w_h^\tau &= 0 && \text{on } \partial Q_{2\epsilon}^- \setminus \Gamma_0, \end{aligned} \right\} \quad (3.35)$$

where $Q_{2\epsilon}^- = \{x \in Q_{2\epsilon} : x_3 < 0\}$.

This problem is uniquely solvable in $((H^2(Q_{2\epsilon}^-))^3)^\perp$, which is the orthogonal

complement in $(L^2(\Omega))^3$ to the constant functions. Moreover w_h^τ satisfies the inequality

$$\|w_h^\tau\|_{H^2(Q_{2\epsilon}^-)} \leq \text{const.} \quad (\|f_h^\tau\|_{L^2} + \|w_h^\tau\|_{H^1}) \leq \text{const.} \|f_h^\tau\|_{L^2}. \quad (3.36)$$

The last inequality holds because of the a priori estimate

$$\|w_h^\tau\|_{H^1} \leq \text{const.} \|f_h^\tau\|_{L^2}.$$

Moreover $\|f_h^\tau\|_{L^2} = \|\Delta_h^\tau(\chi v)\|_{L^2}$ can be estimated by a constant which is independent of h , since $\chi v \in (H^1(Q_{2\epsilon}^-))^3$ and $|h| \leq \epsilon$. This implies that $\exists$ a constant $\tilde{c}$ independent of h with

$$\|w_h^\tau\|_{H^2} \leq \tilde{c}. \quad (3.37)$$

Since

$$\text{grad} : H^2(Q_{2\epsilon}^-) \rightarrow (H^1(Q_{2\epsilon}^-))^3$$

is a continuous operator, (3.37) implies that

$$\|K_h^\tau\|_{H^1} \leq c. \quad (3.38)$$

In order to construct L_h^τ we can follow the very same procedure as above. We define z_h^τ to be the solution of the Dirichlet/Neumann problem

$$\left. \begin{aligned} \Delta z_h^\tau &= g_h^\tau && \text{in } Q_{2\epsilon}^- \\ \frac{\partial z_h^\tau}{\partial n} &= 0 && \text{on } \Gamma_0 \\ z_h^\tau &= 0 && \text{on } \partial Q_{2\epsilon}^- \setminus \Gamma_0, \end{aligned} \right\} \quad (3.39)$$

with $g_h^\tau = \chi \Delta_h^\tau v$ and $\text{grad}(z_h^\tau) = L_h^\tau v$. ■

Theorem 3.10

Let $(u, p) \in J(\Omega_0) \times L^2(\Omega_0)$ be a solution of (3.33), i.e.,

$$\mathcal{A}(v, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2} + \langle \gamma, \varphi \rangle, \quad \forall \varphi \in J(\Omega_0).$$

Furthermore assume that $f \in (L^2(\Omega_0))^3$ and $\gamma \in (H^1(\Omega_0))^3$.

Then:

$$u \in (H^2(\Omega_0))^3 \quad \text{and} \quad p \in (H^1(\Omega_0))^3.$$

In the proof of Theorem 3.10 we shall concentrate on those parts which differ from the standard procedure. For filling in the gaps the reader is referred to [Wlo, §20].

Proof:

Take $\chi \in \mathcal{D}(Q_{3\epsilon})$ and $\varphi \in J$ with support in $Q_{3\epsilon}$. Let us consider the term

$$\mathcal{A}(\Delta_h^\tau(\chi u) - K_h^\tau u, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2}.$$

Then we can proceed as in [Wlo, Satz 20.4] and compute

$$\begin{aligned} \mathcal{A}(\Delta_h^\tau(\chi u) - K_h^\tau u, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} &= \\ &= \mathcal{A}(\Delta_h^\tau(\chi u), \varphi) - \mathcal{A}(K_h^\tau u, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} \\ &= -\mathcal{A}(u, \chi \Delta_h^\tau \varphi) + I(u, \varphi) - \mathcal{A}(K_h^\tau u, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} \\ &= -\mathcal{A}(u, \chi \Delta_h^\tau \varphi - L_h^\tau \varphi) + \langle p \nabla g, \chi \Delta_h^\tau \varphi - L_h^\tau \varphi \rangle_{L^2} + \\ &\quad \langle p \nabla g, \varphi - \chi \Delta_h^\tau \varphi - L_h^\tau \varphi \rangle_{L^2} + I(u, \varphi) + \mathcal{A}(u, L_h^\tau \varphi) - \mathcal{A}(K_h^\tau u, \varphi), \end{aligned} \quad (3.40)$$

where the bilinear form I can be computed by using the following properties for Δ_h^τ :

$$(i) \quad \Delta_h^\tau(\varphi \cdot \psi) = (\Delta_h^\tau \varphi) \psi + \varphi(x + h_\tau) \cdot \Delta_h^\tau \psi,$$

$$(ii) \quad \int_{Q_\epsilon} \Delta_h^\tau \tilde{\varphi} dx = 0,$$

if $\tilde{\varphi}$ is a function with $\text{supp } \tilde{\varphi}$ and $\text{supp } \tilde{\varphi}(x + h_\tau) \subset Q_\epsilon$,
where $h_\tau = (0 \dots, h, \dots, 0)$ with an h as τ^{th} component.

Moreover the following estimate holds:

$$|I(u, \varphi)| \leq c_1 \|u\|_{H^1} \|\varphi\|_{H^1}, \quad (3.41)$$

where the constant c_1 is independent of h . Applying Lemma 3.9 in (3.40) we obtain:

$$\begin{aligned} \mathcal{A}(\Delta_h^\tau(\chi u) - K_h^\tau u, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2} &= \\ &= -\langle f, \chi \Delta_h^\tau \varphi - L_h^\tau \varphi \rangle_{L^2} - \langle \gamma, \chi \Delta_h^\tau \varphi - L_h^\tau \varphi \rangle_{L^2(\Gamma_0)} + \\ &\quad \langle p \nabla g, \varphi - \chi \Delta_h^\tau \varphi - L_h^\tau \varphi \rangle_{L^2} + I(u, \varphi) + \mathcal{A}(u, L_h^\tau \varphi) - \mathcal{A}(K_h^\tau u, \varphi). \end{aligned} \quad (3.42)$$

The only term in (3.42) which can not be estimated in a standard way is the boundary integral term $\langle \gamma, \chi \Delta_h^\tau \varphi \rangle_{L^2(\Gamma_0)}$.

$$\begin{aligned}
\langle \gamma, \chi \Delta_h^\tau \varphi \rangle_{L^2(\Gamma_0)} &= \int_{\Gamma_0} \gamma(x', 1) \chi(x', 1) \Delta_h^\tau \varphi(x', 1) dx' = \\
&= \int_0^1 \frac{\partial}{\partial x_3} \left(\int_{\{z=x_3\} \cap Q_\epsilon} \gamma(x', x_3) \chi(x', x_3) \Delta_h^\tau \varphi(x', x_3) dx' \right) dx_3 \\
&= \int_{Q_\epsilon} \frac{\partial}{\partial x_3} (\chi \gamma) \Delta_h^\tau \varphi dx + \int_{Q_\epsilon} (\chi \gamma) \underbrace{\frac{\partial}{\partial x_3} (\Delta_h^\tau \varphi)}_{\Delta_h^\tau \left(\frac{\partial \varphi}{\partial x_3} \right)} dx.
\end{aligned}$$

For the second integral we apply again the properties (i) and (ii) of Δ_h^τ and obtain

$$\langle \gamma, \chi \Delta_h^\tau \varphi \rangle_{L^2(\Gamma_0)} = \int_{Q_\epsilon} \frac{\partial}{\partial x_3} (\chi \gamma) \Delta_h^\tau \varphi dx - \int_{Q_\epsilon} (\Delta_h^\tau (\chi \gamma)) \frac{\partial \varphi}{\partial x_3} (x + h_\tau) dx.$$

Hence we can conclude that

$$|\langle \gamma, \chi \Delta_h^\tau \varphi \rangle_{L^2(\Gamma_0)}| \leq c_1 \|\gamma\|_{H^1} \|\varphi\|_{H^1}. \quad (3.43)$$

This estimate is due to Weidemaier [Wei, Lemma A.1]. Now (3.42) can be estimated by using (3.41) and (3.43) as

$$\begin{aligned}
|\mathcal{A}(\Delta_h^\tau(\chi u) - K_h^\tau u, \varphi) + \langle p \nabla g, \varphi \rangle_{L^2}| &\leq c_2 (\|f\|_{L^2} + \|\gamma\|_{H^1}) \|\varphi\|_{H^1} + \\
&+ c_3 \|u\|_{H^1} \|\varphi\|_{H^1} + c_4 \|p\|_{L^2} \|\varphi\|_{H^1}.
\end{aligned} \quad (3.44)$$

If we take $\varphi = \Delta_h^\tau(\chi u) - K_h^\tau u$, the J -coerciveness of $\mathcal{A}(\cdot, \cdot)$ (see Def. 3.6) and (3.44) lead to the estimate

$$\begin{aligned}
c_5 \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1}^2 &- k \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{L^2}^2 + \\
&c_4 \|p\|_{L^2} \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1} \leq \\
&\leq c_2 (\|f\|_{L^2} + \|\gamma\|_{H^1}) \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1} + \\
&c_3 \|u\|_{H^1} \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1},
\end{aligned}$$

or

$$\begin{aligned}
c_5 \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1}^2 &\leq \\
&\leq c_6 (\|f\|_{L^2} + \|\gamma\|_{H^1} + \|p\|_{L^2} + \|u\|_{H^1}) \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1}.
\end{aligned}$$

In the last estimate we have used Poincaré's inequality. So finally we obtain

$$c_5 \|\Delta_h^\tau(\chi u) - K_h^\tau u\|_{H^1} \leq \text{const.},$$

which together with Lemma 3.9 implies that $\|\Delta_h^T(\chi u)\|_{H^1} \leq \text{const.}$, independent of h . Hence we have shown that all the tangential derivatives of $\partial_k u$, ($k = 1, 2$), are bounded. The same procedure can be carried out for the tangential derivatives of the pressure p . It remains to show that the normal derivatives for u and p are also bounded. To show this we write down the differential equations in Ω_0 .

$$-\left(g^{kl}u_{,l}^i\right)_{,k} + g^{ik}p_{,k} = f^i, \quad (i = 1, 2, 3), \quad (3.45)$$

$$u_{,i}^i = 0. \quad (3.46)$$

If we differentiate (3.46) with respect to x_3 , it follows that

$$\int_{\Omega_0} \left| \frac{\partial^2 u^3}{\partial x^3 \partial x^3} \right|^2 dx < \infty.$$

Using this result in the third equation of (3.45) we conclude that

$$\int_{\Omega_0} \left| \frac{\partial p}{\partial x^3} \right|^2 dx < \infty.$$

Finally the first two equations of (3.45) imply now that

$$\int_{\Omega_0} \left| \frac{\partial^2 u^i}{\partial x^3 \partial x^3} \right|^2 dx < \infty \quad (i = 1, 2).$$

This proves the theorem ■

In the next chapter we shall need a regularity result in the $W^{k,p}$ -spaces for $p > 2$ and in the Hölder spaces $C^{k,\alpha}$. This result can be obtained from the general elliptic theory since the BVP (3.21) satisfies the complementing boundary conditions (see [Sol, §1] and [Agm/Dou/Nir]). We state the result here without proof.

Theorem 3.11

(a)

Let $p \in \mathbf{R}$, $p \geq 2$, $k \in \mathbf{N}$ and $u \in (W^{k+1,p}(\Omega))^3 \cap J(\Omega)$, $p \in W^{k,p}(\Omega)$ be a solution of (3.21). Suppose that Ω is a C^{k+2} -domain, $f \in (W^{k,p}(\Omega))^3$ and $\gamma \in (W^{k+1,p}(\Omega))^3$. Then:

$$u \in (W^{k+2,p}(\Omega))^3 \quad \text{and} \quad p \in W^{k+1,p}(\Omega).$$

Moreover the following estimate holds:

$$\|u\|_{W^{k+2,p}} + \|p\|_{W^{k+1,p}} \leq \text{const.} (\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k+1,p}} + \|u\|_{W^{k+1,p}}).$$

(b)

Let

$$u \in (C^{k+1,\alpha}(\Omega))^3, p \in C^{k,\alpha}(\Omega), \quad 0 < \alpha < 1,$$

be a solution of (3.21) and assume that Ω is a C^{k+3} -domain, $f \in (C^{k,\alpha}(\Omega))^3$ and $\gamma \in (C^{k+1,\alpha}(\Omega))^3$, then

$$u \in (C^{k+2,\alpha}(\Omega))^3, p \in C^{k+1,\alpha}(\Omega)$$

and

$$\|u\|_{C^{k+2,\alpha}} + \|p\|_{C^{k+1,\alpha}} \leq \text{const.} (\|f\|_{C^{k,\alpha}} + \|\gamma\|_{C^{k+1,\alpha}} + \|u\|_{C^{k+1,\alpha}}).$$

Corollary 3.12

Let $f \in (L^6(\Omega))^3$, $\gamma \in (W^{1,6}(\Omega))^3$, then the solution u is an element of $J \cap (W^{2,6}(\Omega))^3$ and we can estimate:

$$\|u\|_{W^{2,6}} \leq \text{const.} (\|f\|_{L^6} + \|\gamma\|_{W^{1,6}}).$$

Proof:

The only assertion which need to be proved is the inequality. By Theorem 3.11 the $W^{2,6}$ -norm of u can be estimated by:

$$\|u\|_{W^{2,6}} \leq \text{const.} (\|f\|_{L^6} + \|\gamma\|_{W^{1,6}} + \|u\|_{W^{1,6}}).$$

Since the imbedding $H^2 \hookrightarrow W^{1,6}$ is continuous, $\exists$ a constant c_1 such that $\|u\|_{W^{1,6}} \leq c_1 \|u\|_{H^2}$. The H^2 -norm of u can be estimated by Theorem 3.11 again:

$$\begin{aligned} \|u\|_{H^2} &\leq c_2 (\|f\|_{L^2} + \|\gamma\|_{H^1} + \|u\|_{H^1}) \\ &\leq c_2 (\|f\|_{L^6} + \|\gamma\|_{W^{1,6}} + \|u\|_{H^1}). \end{aligned}$$

Furthermore for $\|u\|_{H^1}$ an a priori estimate can be derived from the weak formulation of (3.21), which is:

$$\|u\|_{H^1} \leq c_3 (\|f\|_{L^2} + \|\gamma\|_{H^1}).$$

Considering all these estimates together we finally have estimated that

$$\|u\|_{W^{1,\varepsilon}} \leq \text{const.} (\|f\|_{L^\varepsilon} + \|\gamma\|_{W^{1,\varepsilon}}).$$

This estimate together with the first one proves the assertion. ■

Chapter 4

Fixed point formulation

So far we have transformed the free boundary problem (1.53 and 1.54) into an integral equation (see Theorem 2.33) on Ω and a differential equation on Γ_b (see the normal stress condition in (1.54)). Those equations are in principal of the following form:

$$\frac{1}{2}E(u, \varphi) + \mu_1 \langle L_1 u, \varphi \rangle_{L^2} + \mu_2 \langle L_2 u, \varphi \rangle_{L^2} = \lambda \{ \langle f, \varphi \rangle_{L^2} + \langle \gamma, \varphi \rangle_{L^2(\Gamma)} \}, \quad (4.1)$$

$\forall \varphi \in J,$

$$\operatorname{div} \left(\frac{\nabla \zeta}{(1 + |\nabla \zeta|^2)^{1/2}} \right) + \frac{\lambda}{\sigma} \beta_0(\lambda, \zeta) = \frac{\lambda}{\sigma} g, \quad \text{on } \Gamma_b, \quad (4.2)$$

where $f = f_1 + f_2$ with

$$f_1 \in (C^{0,\alpha}(\Omega))^3 \quad \text{and} \quad f_2 \in (H^1(\Omega))^3,$$

$$\gamma = \gamma_1 \tau_1 + \gamma_2 \tau_2 \in (C^{1,\alpha}(\Omega))^3,$$

and

$$g \in C^{1,2\alpha}(\Gamma_b) \quad \text{for some } \alpha, 0 < \alpha \leq 1/4.$$

It has been shown in Theorem 3.5 that the operator

$$\frac{1}{2}E(\cdot, \varphi) + \mu_1 \langle L_1 \cdot, \varphi \rangle_{L^2} + \mu_2 \langle L_2 \cdot, \varphi \rangle_{L^2}$$

defines a solution operator

$$T_\Omega : (C^{0,\alpha}(\Omega))^3 \times (C^{1,\alpha}(\Omega))^3 \rightarrow (W^{2,p}(\Omega))^3, \quad 2 \leq p \leq 6, \quad (4.3)$$

$$(f, \gamma) \mapsto u = T_\Omega(f, \gamma). \quad (4.4)$$

Here we have assumed for simplicity that $f_2 = 0$.

Since

$$(C^{0,\alpha}(\Omega))^3 \times (C^{1,\alpha}(\Omega))^3 \xrightarrow{\text{compactly}} (L^p(\Omega))^3 \times (W^{1,p}(\Omega))^3, \quad \text{for } p < \infty,$$

and

$$(H^1(\Omega))^3 \xrightarrow{\text{compactly}} (L^p(\Omega))^3 \times (W^{1,p}(\Omega))^3, \quad \text{for } 2 \leq p < 6,$$

T_Ω is a compact operator into $(W^{2,p}(\Omega))^3$, $2 \leq p < 6$.

The equation (4.2) can also be written in the form

$$\begin{aligned} & \frac{1}{(1 + |\nabla \zeta|^2)^{1/2}} \Delta \zeta - \sum_{i,j} \frac{\partial_i \zeta \partial_j \zeta}{(1 + |\nabla \zeta|^2)^{3/2}} \partial_{ij} \zeta - \frac{2}{\sigma} \frac{\sum_k \partial_k \zeta}{(1 + \zeta)(1 + |\nabla \zeta|^2)} \partial_{11} \zeta \\ & + 2 \frac{\lambda}{\sigma} \frac{\sum_k \partial_k \zeta}{(1 + \zeta)(1 + |\nabla \zeta|^2)} (\mu_1 \partial_{11} \zeta + \mu_2 \partial_{22} \zeta) = \frac{\lambda}{\sigma} g, \quad \text{in } \Gamma_b. \end{aligned} \quad (4.5)$$

Let us introduce operators $\mathcal{M}_1(\xi)$ and $\mathcal{M}_2(\xi)$ for $\xi \in C^{3,\alpha}(\Gamma_b)$:

Definition 4.1

For $\xi \in C^{3,\alpha}(\Gamma_b)$, we define:

$$a_0(\xi) \stackrel{\text{def}}{=} \frac{1}{(1 + |\nabla \xi|^2)^{1/2}}, \quad (4.6)$$

$$a_{ij}(\xi) \stackrel{\text{def}}{=} \frac{\partial_i \xi \partial_j \xi}{(1 + |\nabla \xi|^2)^{3/2}} + \frac{2}{\sigma} \frac{(\sum_k \partial_k \xi) \delta_{i1} \delta_{j1}}{(1 + \xi)(1 + |\nabla \xi|^2)}, \quad (4.7)$$

$$b_i(\xi) \stackrel{\text{def}}{=} 2\mu_i \frac{\sum_k \partial_k \xi}{(1 + \xi)(1 + |\nabla \xi|^2)}, \quad (4.8)$$

and

$$\mathcal{M}_1(\xi) \zeta \stackrel{\text{def}}{=} a_0 \Delta \zeta - \sum_{i,j} a_{ij} \partial_{ij} \zeta, \quad (4.9)$$

$$\mathcal{M}_2(\xi) \zeta \stackrel{\text{def}}{=} \sum_i b_i \partial_{ii} \zeta. \quad (4.10)$$

Remark 4.2

Let μ_1, μ_2, σ and $\Lambda_0 \in \mathbf{R}$ be fixed and $\lambda \in [0, \Lambda_0]$. Then we can find a small $\rho > 0$ such that for $\xi \in B_\rho(0) \subset C^{3,\alpha}(\Gamma_b)$, $\mathcal{M}_1(\xi)$ is a uniformly elliptic operator, since $a_0(\xi)$ is the only coefficient of order zero with respect to ξ .

Instead of considering (4.5) we look at the equation

$$\mathcal{M}_1(\xi)\zeta + \frac{\lambda}{\sigma}\mathcal{M}_2(\xi)\zeta = \frac{\lambda}{\sigma}g, \quad \text{in } \Gamma_b, \quad (4.11)$$

where $g \in C^{1,2\alpha}(\Gamma_b)$.

It will be shown that $\mathcal{M}_1(\xi)$ subject to periodic boundary conditions can be inverted as long as $\xi \in B_\rho(0) \subset C^{3,\alpha}(\Gamma_b)$. The inverse operator $T_\Gamma(\xi) : C^{1,2\alpha}(\Gamma_b) \rightarrow C^{3,\alpha}(\Gamma_b)$ is moreover compact. In order to deal with (4.1) and (4.2) at the same time we need the compactness of T_Ω and T_Γ also with respect to ζ and $\xi \in C^{3,\alpha}(\Gamma_b)$, respectively. The proofs of these assertions will be the main issue in this paragraph. Once those results are established we are ready to formulate the fixed point equation.

4.1 Compactness of T_Ω

T_Ω depends on ζ via f, γ and the domain Ω . In order to examine the dependence on ζ we transform the problem onto Ω_0 , where we define the operator T_{Ω_0} . For the transformation let us recall some definitions from Section 1.3:

The diffeomorphisms $\Phi : \Omega_0 \rightarrow \Omega$ and $\Psi = \Phi^{-1} : \Omega \rightarrow \Omega_0$ (see (1.12) and (1.19), respectively) are determined by the doubly periodic function $\zeta \in C^{3,\alpha}(\Gamma_b)$. With this notation we can make the following definition:

Definition 4.3

Let $\zeta \in C^{3,\alpha}(\Gamma_b)$ be a periodic function and

$$f \in (W^{m,p}(\Omega_0))^3, m \in \{0, 1, 2\}, p \geq 2.$$

Let $\Phi : \Omega_0 \rightarrow \Omega$ and $\Psi : \Omega \rightarrow \Omega_0$, given by (1.12) and (1.19), be the diffeomorphisms which are determined by ζ . Then we define the transformation mapping

$$\pi(\zeta) : (W^{m,p}(\Omega_0))^3 \rightarrow (W^{m,p}(\Omega))^3$$

by

$$f \mapsto \sqrt{h}(D\Phi)^t(\Psi(x)) \cdot f(\Psi(x)).$$

Here h is the determinant of the metric tensor h_{ij} (see (1.23)). Moreover we call the usual transformation for covariant vectors $\hat{\pi}(\zeta)$ which is given by

$$\hat{\pi}(\zeta) : (W^{m,p}(\Omega_0))^3 \rightarrow (W^{m,p}(\Omega))^3,$$

$$g \mapsto (D\Psi) \cdot f(\Psi(x)).$$

Remark 4.4

The transformation mapping $\pi(\zeta)$ is well defined since the regularity of $f \in (W^{m,p}(\Omega_0))^3$, $m \leq 2$, does not get lowered by multiplication with C^2 -functions. We have used this transformation before and we know from the discussion below (1.42) that $\pi(\zeta)$ is moreover a mapping between

$$(W^{m,p}(\Omega_0))^3 \cap J(\Omega_0) \rightarrow (W^{m,p}(\Omega))^3 \cap J(\Omega).$$

This means that $v \in (W^{m,p}(\Omega_0))^3$ with $\operatorname{div} v = 0$ is mapped by $\pi(\zeta)$ into $\pi(\zeta)v$ with $\operatorname{div}(\pi(\zeta)v) = 0$ in the coordinates of Ω .

Definition 4.5

Let $\Omega \subset \mathbb{R}^3$ be a domain with the upper surface given as the graph of the function $1 + \zeta$, where $\zeta \in C^{3,\alpha}(\Gamma_b)$ is a doubly periodic function. Then we define

$$T_{\Omega_0}(\zeta) : (W^{m,p}(\Omega_0))^3 \rightarrow (W^{m,p}(\Omega_0))^3, \quad m \in \{0, 1, 2\}, p \geq 2,$$

by

$$\begin{aligned} T_{\Omega_0}(\zeta) &\stackrel{\text{def}}{=} \pi(\zeta)^{-1} \circ T_{\Omega} \circ \pi(\zeta), \\ f &\mapsto T_{\Omega_0}(\zeta)f = \pi(\zeta)^{-1} \circ T_{\Omega}(\pi(\zeta)f). \end{aligned}$$

More precisely, the function ζ is an element of the space $V_{\Gamma, C^{3,\alpha}}$, which we define now.

Definition 4.6

Define the space $V_{\Gamma, C^{3,\alpha}}$ in the following way:

$$\begin{aligned} \dot{V}_{\Gamma} &\stackrel{\text{def}}{=} \{ \varphi \in C^{\infty}(\Gamma_b) : \partial^{\alpha} \varphi \text{ is periodic in } x \text{ and } y \forall |\alpha|; \int_{\Gamma_b} \varphi \, ds = 0 \}, \\ V_{\Gamma, C^{k,\alpha}} &\stackrel{\text{def}}{=} \overline{\dot{V}_{\Gamma}}^{\|\cdot\|_{C^{k,\alpha}}}, \quad k \in \mathbb{N}, \alpha > 0, \end{aligned}$$

where we have taken the closure of $\dot{V}_{\Gamma}$ in $C^{k,\alpha}(\Gamma_b)$.

Now we can state the compactness result for T_{Ω_0} :

Theorem 4.7

Let $(\zeta_l)_{l \in \mathbb{N}} \subset V_{\Gamma, C^{3,\alpha}}$, $(f_m)_{m \in \mathbb{N}} \subset (C^{0,\alpha}(\Omega_0))^3$ and $(\gamma_n)_{n \in \mathbb{N}} \subset (C^{1,\alpha}(\Omega_0))^3$

be bounded sequences in their respective spaces. Furthermore let us assume that $(\zeta_l)_{l \in \mathbb{N}} \subset B_{1/2}(0)$, the ball of radius $1/2$ in $C^{3,\alpha}$.

Then the sequence $u_{lmn} = T_{\Omega_0}(\zeta_l, f_m, \gamma_n)$ contains a convergent subsequence in $(H^2(\Omega_0))^3$.

Proof:

By compact imbeddings of the three spaces and passing to a subsequence if necessary we can assume without loss of generality that the sequences $(\zeta_l)_{l \in \mathbb{N}} \subset V_{\Gamma, C^3}$, $(f_m)_{m \in \mathbb{N}} \subset (L^2(\Omega_0))^3$ and $(\gamma_n)_{n \in \mathbb{N}} \subset (H^1(\Omega_0))^3$ are already convergent. Let us say that

$$(\zeta_n, f_n, \gamma_n) \xrightarrow{n \rightarrow \infty} (\zeta_0, f_0, \gamma_0).$$

Let $T_{\Omega_0}(\zeta_0, f_0, \gamma_0) = u_0$. Then the following lemma proves the theorem.

Lemma 4.8

Let $(\zeta_0, f_0, \gamma_0) \in X$ be given, where

$$X \stackrel{\text{def}}{=} \left(V_{\Gamma, C^3} \cap B_{1/2}(0) \right) \times (L^2(\Omega_0))^3 \times (H^1(\Omega_0))^3$$

and

$$u_0 = T_{\Omega_0}(\zeta_0, f_0, \gamma_0) \in (H^2(\Omega_0))^3 \cap J(\Omega_0).$$

Then there exist $\rho > 0$, $\delta > 0$ and a continuous map

$$S : B_\rho(\zeta_0, f_0, \gamma_0) \subset X \rightarrow B_\delta(u_0) \subset (H^2(\Omega_0))^3$$

such that

$$u = S(\zeta, f, \gamma) = T_{\Omega_0}(\zeta, f, \gamma), \quad \forall (\zeta, f, \gamma) \in B_\delta(\zeta_0, f_0, \gamma_0).$$

Proof:

By assumption

$$u = T_{\Omega_0}(\zeta, f, \gamma) = \pi(\zeta)^{-1} \circ T_\Omega \circ \pi(\zeta)(f, \gamma)$$

or

$$\tilde{u} = T_\Omega(\tilde{f}, \tilde{\gamma}), \tag{4.12}$$

where

$$\tilde{u} = \pi(\zeta)u \quad \text{and} \quad (\tilde{f}, \tilde{\gamma}) = \pi(\zeta)(f, \gamma) \stackrel{\text{def}}{=} (\pi(\zeta)f, \hat{\pi}(\zeta)\gamma).$$

Equation (4.12) is equivalent to

$$\frac{1}{2}E(\tilde{u}, \varphi) + \mu_1 \langle L_1 \tilde{u}, \varphi \rangle_{L^2} + \mu_2 \langle L_2 \tilde{u}, \varphi \rangle_{L^2} = \langle \tilde{f}, \varphi \rangle_{L^2} + \langle \tilde{\gamma}, \varphi \rangle_{L^2(\Gamma)}, \quad (4.13)$$

$$\forall \varphi \in J(\Omega),$$

which is equivalent to

$$\left. \begin{aligned} -\Delta \tilde{u} + \text{grad } p + \mu_1 L_1 \tilde{u} + \mu_2 L_2 \tilde{u} &= \tilde{f} \quad \text{in } \Omega \\ \tau_k \cdot t(\tilde{u}) \cdot n|_{\Gamma} &= \tilde{\gamma}, \quad (k=1,2), \\ \tilde{u} &\in J(\Omega). \end{aligned} \right\} \quad (4.14)$$

In (4.14) the function $p \in H^1(\Omega)$ is uniquely determined up to a constant by f, ζ and u . The BVP (4.14) can be transformed to a BVP on Ω_0 (see (3.23)).

$$\left. \begin{aligned} -\left(g^{kl}\hat{u}^i_{,l}\right)_{,k} + g^{ik}\hat{p}_{,k} + \mu_1 L_1 \hat{u}^i + \mu_2 L_2 \hat{u}^i - \hat{f}^i &= 0 \\ \hat{u}^i_{,i} &= 0 \end{aligned} \right\} \text{ in } \Omega_0$$

$$\left. \begin{aligned} \tau_k t^{ik}(\hat{u}) n_i|_{\Gamma_0} &= \hat{\gamma}_k, \quad (k=1,2), \\ \hat{u} &\in V, \end{aligned} \right\} \quad (4.15)$$

where

$$\left. \begin{aligned} u &= \pi(\zeta)^{-1} \tilde{u} = \sqrt{g} \hat{u}, \\ f &= \pi(\zeta)^{-1} \tilde{f} = \sqrt{g} \hat{f}, \\ \gamma &= \hat{\pi}(\zeta)^{-1} \tilde{\gamma} = \hat{\gamma}. \end{aligned} \right\} \quad (4.16)$$

By (4.16) it is enough to find $\rho, \delta > 0$ and a continuous map

$$\hat{S} : B_\rho(\zeta_0, \hat{f}_0, \hat{\gamma}_0) \subset X \rightarrow B_\delta(\hat{u}_0) \subset (H^2(\Omega_0))^3$$

such that $\hat{u} = \hat{S}(\zeta, \hat{f}, \hat{\gamma})$ is a solution of (4.15). This can be done by the implicit function theorem in Banach spaces (see e.g. [Dei, Th. 15.1]). The system (4.15) is an equation of the form $F(\zeta, \hat{f}, \hat{\gamma}, \hat{p}, \hat{u}) = 0$, where F is a mapping between $X \times Y \rightarrow Z$ with

$$Y = \left(H^1(\Omega_0)\right)^\perp \times \left((H^2(\Omega_0))^3 \cap J(\Omega_0)\right),$$

and

$$Z = (L^2(\Omega_0))^3 \times H^1(\Omega_0) \times (L^2(\Gamma_b))^2.$$

We have called here the complement of the constant functions in the L^2 - duality $(H^1(\Omega_0))^\perp$. One can see from (4.15) that F is continuous with respect to ζ , $\hat{f}$ and $\hat{\gamma}$ and differentiable with respect to $\hat{u}$ and $\hat{p}$. The differentiability of F is obvious, since it is linear in $\hat{u}$ and $\hat{p}$. Moreover $(\hat{u}, \hat{p}) \in Y$ is the unique solution of (4.15) for given $(\zeta, \hat{f}, \hat{\gamma})$ (see Theorem 3.5) and therefore the Fréchet derivative $D_{(\hat{p}, \hat{u})}F$ is an invertible operator at any given point. Now we can apply the implicit function theorem by which there exist $\rho, \delta > 0$ and a continuous map

$$\hat{R} : B_\rho(\zeta_0, \hat{f}_0, \hat{\gamma}_0) \subset X \rightarrow B_\delta(\hat{p}_0, \hat{u}_0) \subset Y$$

such that

$$F(\zeta, \hat{f}, \hat{\gamma}, \hat{R}(\zeta, \hat{f}, \hat{\gamma})) = 0 \quad \text{in } B_\rho(\zeta_0, \hat{f}_0, \hat{\gamma}_0).$$

Define $\hat{S} \stackrel{\text{def}}{=} P \circ \hat{R}$ where $P : Y \rightarrow Y$ is the projection onto $(H^2(\Omega_0))^3 \cap J$. ■

4.2 Compactness of T_Γ

We first show that an inverse operator can be defined for the operator $\mathcal{M}_1(\xi)$ (see Def. 4.1) when subject to periodic boundary conditions. Let us consider the following boundary value problem:

$$\left. \begin{aligned} \mathcal{M}_1(\xi)\zeta &= g \in V_{\Gamma, C^{1,2\alpha}} \\ \zeta &\in V_{\Gamma, C^{3,\alpha}} \end{aligned} \right\} \quad (4.17)$$

Lemma 4.9

Let $\xi \in V_{\Gamma, C^{3,\alpha}}$ with $\|\xi\|_{C^{3,\alpha}} \leq \rho$ and $g \in V_{\Gamma, C^{1,2\alpha}}$. If $\rho > 0$ is small enough, then $\exists!$ solution ζ of (4.17).

Proof:

The operator $\mathcal{M}_1(\xi)$ can be written as:

$$\begin{aligned} \mathcal{M}_1(\xi) &= \frac{1}{(1 + |\nabla \xi|^2)^{1/2}} \Delta \zeta - \sum_{i,j} a_{ij}(\xi) \partial_{ij} \zeta \\ &= \Delta \zeta + \left(\frac{1}{(1 + |\nabla \xi|^2)^{1/2}} - 1 \right) \Delta \zeta - \sum_{i,j} a_{ij}(\xi) \partial_{ij} \zeta. \end{aligned}$$

Define:

$$\hat{a}_{ij}(\xi) \stackrel{\text{def}}{=} \begin{cases} a_{ij}(\xi), & i \neq j \\ (1 + |\nabla \xi|^2)^{-1/2} - 1 - a_{ii}(\xi), & i = j \end{cases},$$

and

$$\hat{\mathcal{M}}_1(\xi)\zeta \stackrel{def}{=} \sum_{i,j} \hat{a}_{ij}(\xi) \partial_{ij} \zeta.$$

Furthermore let

$$\mathcal{M}_t(\xi)\zeta \stackrel{def}{=} \Delta \zeta + t \hat{\mathcal{M}}(\xi)\zeta.$$

It is clear from the construction that $\mathcal{M}_0\zeta = \Delta\zeta$. Moreover, there exists a unique solution $\zeta_0 \in V_{\Gamma, C^{3,\alpha}}$ for the BVP

$$\mathcal{M}_0\zeta = \Delta\zeta = g. \quad (4.18)$$

This can be seen in the following way:

Uniqueness

Take $g = 0$ and suppose that $\zeta \in V_{\Gamma, C^{3,\alpha}}$ is a solution of (4.17), i.e.: $\Delta\zeta = 0$. We multiply this equation with ζ , integrate over Γ_b and apply Gauß' theorem to it. This leads to $\int_{\Gamma_b} |\nabla\zeta|^2 ds = 0$, which implies that $|\nabla\zeta| = 0$ a.e. . Hence $\zeta_0 \equiv \text{const.}$. But the only constant in $V_{\Gamma, C^{3,\alpha}}$ is zero.

Existence

We want to solve the problem first in the space V_{Γ, H^1} , which is the closure of $\dot{V}_{\Gamma}$ in the H^1 - norm. Since the Poincaré inequality holds in V_{Γ, H^1} , $\langle \nabla\zeta, \nabla\zeta \rangle_{L^2}$ is an inner product in V_{Γ, H^1} . Moreover this inner product induces a norm equivalent to the H^1 - norm. The weak formulation of (4.17) is

$$-\int_{\Gamma_b} \nabla\zeta \cdot \nabla\varphi ds = \langle g, \varphi \rangle_{L^2}, \quad \forall \varphi \in V_{\Gamma, H^1}. \quad (4.19)$$

By the Riesz representation theorem $\exists!$ solution $\zeta_0 \in V_{\Gamma, H^1}$, which is by regularity considerations in $H^2(\Gamma_b)$. Hence by the Sobolev imbedding

$$H^2 \hookrightarrow C^{0,\beta}, \quad \forall \beta \text{ with } 0 \leq \beta < 1,$$

we obtain

$$\zeta_0 \in V_{\Gamma, C^{0,\beta}}.$$

Extending ζ_0 and g smoothly to $\mathbf{R}^2$ we can solve the Dirichlet problem

$$\begin{aligned} \Delta\zeta &= g, & \text{in } \Gamma_e, \\ \zeta &= \zeta_0, & \text{on } \partial\Gamma_e, \end{aligned} \quad (4.20)$$

where $\Gamma_e \supset \Gamma_b$ is a smooth domain. Then the solution ζ of (4.20) is in the space $C^{0,\beta}(\bar{\Gamma}_e) \cap C^{3,2\alpha}(\overset{\circ}{\Gamma}_e)$. Hence we have obtained a solution $\zeta \in V_{\Gamma,C^{3,2\alpha}}$. Therefore there exists a compact solution operator

$$\begin{aligned} K_0 : V_{\Gamma,C^{1,2\alpha}} &\rightarrow V_{\Gamma,C^{3,\alpha}} \\ g &\mapsto K_0 g = \zeta_0. \end{aligned}$$

Now we want to solve $\mathcal{M}_t \zeta = \Delta \zeta + t \hat{\mathcal{M}}_1 \zeta = g$, which is equivalent to $\zeta + t K_0 \hat{\mathcal{M}}_1 \zeta = K_0 g$. Define $c_0 \stackrel{\text{def}}{=} \|K_0 \hat{\mathcal{M}}_1\|_{\mathcal{L}(V_{\Gamma,C^{3,2\alpha}})}$ and $\delta = \frac{1}{2c_0}$, where $\mathcal{L}(V_{\Gamma,C^{3,2\alpha}})$ is the space of the linear bounded operators from $V_{\Gamma,C^{3,2\alpha}}$ into itself. Then $I + t K_0 \hat{\mathcal{M}}_1$ can be inverted $\forall t \in [0, \delta]$. Hence the operator $K_t \stackrel{\text{def}}{=} (I + t K_0 \hat{\mathcal{M}}_1)^{-1} K_0$ can be defined which is the solution operator to the problem

$$\begin{aligned} \mathcal{M}_t \zeta &= g \\ \zeta &\in V_{\Gamma,C^{3,\alpha}}. \end{aligned}$$

Moreover, $K_t : V_{\Gamma,C^{1,2\alpha}} \rightarrow V_{\Gamma,C^{3,\alpha}}$ is compact. Since the norm of $\hat{\mathcal{M}}_1$ depends on the norm of ξ at least linearly, we can assume that we have chosen $\|\xi\|_{C^{3,\alpha}}$ so small that $\delta \geq 1$. Hence we have constructed solution operators $K_t \forall t \in [0, 1]$,

$$K_t : V_{\Gamma,C^{1,2\alpha}} \rightarrow V_{\Gamma,C^{3,\alpha}},$$

which are, moreover, compact. ■

According to this Lemma we can define a solution operator T_Γ for (4.17).

Definition 4.10

The solution operator for (4.17), called T_Γ , is defined by:

$$T_\Gamma : (V_{\Gamma,C^{k+2,\alpha}} \cap B_\rho(0)) \times V_{\Gamma,C^{k,2\alpha}} \rightarrow V_{\Gamma,C^{k+2,2\alpha}}$$

by

$$(\xi, g) \mapsto \zeta = T_\Gamma(\xi, g) \stackrel{\text{def}}{=} T_\Gamma(\xi)g.$$

The operator T_Γ is compact with respect to g for fixed ξ , when T_Γ is considered as a mapping into $V_{\Gamma,C^{k+2,\alpha}}$. In the next lemma it will be shown that T_Γ is a bounded mapping with respect to ξ and g .

Lemma 4.11

Let $B_\rho \subset V_{\Gamma, C^{k+2, \alpha}}$, ρ small enough, and $B_R \subset V_{\Gamma, C^{k, 2\alpha}}$, R a fixed number, be balls with radii ρ and R , respectively. Then:

$$T_\Gamma : B_\rho \times B_R \rightarrow V_{\Gamma, C^{k+2, 2\alpha}}$$

is a bounded operator.

Corollary 4.12

Choose B_ρ and B_R as in Lemma 4.11. Then

$$T_\Gamma : B_\rho \times B_R \rightarrow V_{\Gamma, C^{k+2, \alpha}}$$

is a compact operator.

Proof of Lemma 4.11:

For $\xi \in B_\rho \subset V_{\Gamma, C^{k+2, \alpha}}$ and $g \in V_{\Gamma, C^{k, 2\alpha}}$ given, we shall derive first an H^1 -estimate for ζ . Once this estimate is established, the assertion follows by standard regularity estimates. Multiply (4.17) by $\varphi \in V_{\Gamma, H^1}$ and integrate over Γ_b :

$$\int_{\Gamma_b} \mathcal{M}_1(\xi) \zeta \varphi \, ds = \int_{\Gamma_b} g \varphi \, ds, \quad (4.21)$$

where

$$\begin{aligned} \int_{\Gamma_b} \mathcal{M}_1(\xi) \zeta \varphi \, ds &= \int_{\Gamma_b} \{a_0(\xi) \Delta \zeta - \sum_{i,j} a_{ij}(\xi) \partial_{ij} \zeta\} \varphi \, ds \\ &= - \sum_{i,j} \int_{\Gamma_b} [a_0(\xi) \delta_{ij} - a_{ij}(\xi)] \partial_i \zeta \partial_j \varphi \, ds \\ &\quad + \sum_{i,j} \int_{\Gamma_b} \partial_j (a_0(\xi) \delta_{ij} - a_{ij}(\xi)) \partial_i \zeta \varphi \, ds. \end{aligned}$$

Then (4.21) becomes:

$$\begin{aligned} \sum_{i,j} \int_{\Gamma_b} [a_0(\xi) \delta_{ij} - a_{ij}(\xi)] \partial_i \zeta \partial_j \varphi \, ds = \\ - \int_{\Gamma_b} g \varphi \, ds + \sum_{i,j} \int_{\Gamma_b} \partial_j (a_0(\xi) \delta_{ij} - a_{ij}(\xi)) \partial_i \zeta \varphi \, ds. \end{aligned} \quad (4.22)$$

Take $\varphi = \zeta$. Since $\mathcal{M}_1(\xi)$ is uniformly elliptic, $\exists$ a constant $c_0 > 0$ such that

$$\sum_{i,j} [a_{0i}(\xi)\delta_{ij} - a_{ij}(\xi)]\partial_i\zeta\partial_j\zeta \geq c_0\|\nabla\zeta\|_{L^2}^2.$$

Now (4.23) can be estimated by:

$$c_0\|\nabla\zeta\|_{L^2}^2 \leq \|g\|_{L^2}\|\zeta\|_{L^2} + c_1\|\xi\|_{C^{k+2,\alpha}}\|\nabla\zeta\|_{L^2}\|\zeta\|_{L^2}, \quad (4.23)$$

with $c_1 > 0$. Since V_{Γ,H^1} is orthogonal to the constant functions, Poincaré's inequality $\|\zeta\|_{L^2} \leq c_p\|\nabla\zeta\|_{L^2}$ holds. Then (4.23) can be further estimated by:

$$c_0\|\nabla\zeta\|_{L^2} \leq c_p\|g\|_{L^2} + c_1c_p\|\xi\|_{C^{k+2,\alpha}}\|\nabla\zeta\|_{L^2}$$

i.e.,

$$(c_0 - c_1c_p\|\xi\|_{C^{k+2,\alpha}})\|\nabla\zeta\|_{L^2} \leq c_p\|g\|_{L^2}. \quad (4.24)$$

Now choose $\rho > 0$ so small that $z := \frac{c_1c_p\rho}{c_0} < 1/2$. Hence we obtain for the estimate (4.24)

$$\begin{aligned} \|\nabla\zeta\|_{L^2} &\leq \frac{c_p}{c_0}\|g\|_{L^2} + 2\frac{c_1c_p^2}{c_0}\|g\|_{L^2}\|\xi\|_{C^{k+2,\alpha}} \\ &\leq \frac{c_p}{c_0}\|g\|_{L^2} + 2\frac{c_1c_p^2}{c_0}R \operatorname{vol}(\Gamma_b)\|\xi\|_{C^{k+2,\alpha}}, \end{aligned} \quad (4.25)$$

or

$$\|\nabla\zeta\|_{L^2} \leq \operatorname{const.} (\|g\|_{L^2} + \|\xi\|_{C^{k+2,\alpha}}). \quad (4.26)$$

Since g and ζ are periodic, a solution ζ of (4.17) is also a solution on a bigger domain, for example the one we get by gluing copies of Γ_b to the sides of Γ_b , such that Γ_b is completely in the interior of the new big domain, which we might call $\hat{\Gamma}_b$. Let Γ_e be a smooth domain with $\hat{\Gamma}_b \subset \Gamma_e \subset \bar{\Gamma}_e \subset \hat{\Gamma}_b$. Define $\varphi_0 \stackrel{\text{def}}{=} \zeta|_{\Gamma_e}$. Then ζ is the solution of the Dirichlet problem

$$\begin{aligned} \mathcal{M}_1(\xi)\zeta &= g \quad \text{in } \Gamma_e, \\ \zeta &= \varphi_0 \quad \text{on } \partial\Gamma_e. \end{aligned}$$

So in order to obtain H^2 - and $C^{k+2,2\alpha}$ - estimates for ζ we can use the interior estimates for the Dirichlet problem (see e.g. [Gil/Tru, Lemma 6.5]). Then (4.26) implies:

$$\begin{aligned} \|\zeta\|_{H^2} &\leq \operatorname{const.} (\|g\|_{L^2} + \|\zeta\|_{H^1}) \\ &\leq \operatorname{const.} (\|g\|_{C^{k,2\alpha}} + \|\xi\|_{C^{k+2,\alpha}}). \end{aligned} \quad (4.27)$$

Now we might use the continuous imbedding $H^2 \hookrightarrow C^{0,\beta}$, $0 < \beta < 1$, (see e.g. [Adm, Th. 5.4]), and the interior Schauder estimate (maybe several times depending on k) and we obtain:

$$\|\zeta\|_{C^{k+2,2\alpha}} \leq \text{const.} (\|g\|_{C^{k,2\alpha}} + \|\xi\|_{C^{k+2,\alpha}}). \quad (4.28)$$

■

The question arises, under which conditions on g the problem

$$\left. \begin{aligned} \mathcal{M}_1(\zeta)\zeta &= g \in V_{\Gamma, C^{1,2\alpha}} \\ \zeta &\in V_{\Gamma, C^{3,\alpha}} \end{aligned} \right\} \quad (4.29)$$

will have a solution. The solution of (4.29) is a fixed point of the mapping $\zeta \mapsto T_{\Gamma}(\zeta)g$, if $\|\zeta\|_{C^{k+2,\alpha}} < \rho$. This requires that $\|g\|_{C^{k,2\alpha}}$ has to be small.

Lemma 4.13

Let $\xi \in B_{\rho}(0) \subset V_{\Gamma, C^{k+2,\alpha}}$, $\rho > 0$ small enough and fixed. Then there exists a constant $b_0 > 0$, depending on ρ , such that:

If

$$g \in V_{\Gamma, C^{k,2\alpha}} \text{ with } \|g\|_{C^{k,2\alpha}} < b_0^{-1},$$

then

$$\zeta = T_{\Gamma}(\xi, g) \in B_{\rho}(0) \subset V_{\Gamma, C^{k+2,\alpha}}.$$

Proof:

Using (4.24) we can estimate $\nabla \zeta$ by

$$\|\nabla \zeta\|_{H^1} \leq \tilde{b}_0 \|g\|_{C^{k,2\alpha}} \rho, \quad (4.30)$$

$$\text{with } \tilde{b}_0 = \frac{c_p}{c_0} (\rho^{-1} + 2c_1 c_p).$$

From (4.30) we can proceed as in the proof of Lemma 4.11 to reach a $C^{k+2,\alpha}$ -estimate for ζ , which is now of the form

$$\|\zeta\|_{C^{k+2,\alpha}} \leq \text{const.} \|g\|_{C^{k+2,\alpha}} \rho \quad (4.31)$$

with the constant equal to b_0 .

■

4.3 The fixed point formulation

Now we are going back to the full problem. We shall write it in the following form:

$$\frac{1}{2}E(u, \varphi) + \mu_1 \langle L_1 u, \varphi \rangle_{L^2} + \mu_2 \langle L_2 u, \varphi \rangle_{L^2} = \lambda \left\{ F(\lambda, u, \zeta) + \langle \gamma, \varphi \rangle_{L^2(\Gamma)} \right\} \quad (4.32)$$

$$\mathcal{M}_1(\zeta)\zeta + \frac{\lambda}{\sigma}\mathcal{M}_2(\zeta)\zeta = \frac{\lambda}{\sigma}G(\lambda, u, \zeta), \quad (4.33)$$

where

$$F(\lambda, u, \zeta) = f(\lambda, \zeta) - L_0(\lambda, \zeta)u - Bu, \quad (4.34)$$

$$G(\lambda, u, \zeta) = \beta_1(\lambda, \zeta) + n(\zeta) \cdot t(u) \cdot n(\zeta) |_{\Gamma}, \quad (4.35)$$

and γ_1, γ_2 are defined by (3.13) and (3.14), respectively. The function β_1 (see Def. 1.2) is given in dimensionless form as

$$\beta_1 = 2 \frac{(1 - \frac{\mu_1}{\lambda})\zeta_x - \frac{\mu_2}{\lambda}\zeta_y}{(1 + \zeta)^2}. \quad (4.36)$$

Moreover, L_0 (3.17) and B are defined by

$$L_0 u = \sum_k u_{0k}^0 \partial_k u + \sum_k u_k \partial_k u_0^0$$

and

$$Bu = \sum_k u^k \partial_k u.$$

Applying the inverse operators T_{Ω_0} and T_{Γ} to the equations (4.32) and (4.33) we obtain:

$$u = \lambda T_{\Omega_0}(\zeta, F(\lambda, u, \zeta), \gamma), \quad (4.37)$$

$$\left(I + \frac{\lambda}{\sigma} T_{\Gamma}(\zeta) \mathcal{M}_2(\zeta) \right) \zeta = \frac{\lambda}{\sigma} (G(\lambda, u, \zeta)). \quad (4.38)$$

In case the norm of the operator $\frac{\lambda}{\sigma} T_{\Gamma}(\zeta) \mathcal{M}_2(\zeta)$ is less than one, the operator on the left hand side of (4.38) can be inverted. Since $\lambda \leq \Lambda_0$ we might obtain $\| \frac{\lambda}{\sigma} T_{\Gamma}(\zeta) \mathcal{M}_2(\zeta) \|_{\mathcal{L}(C^{3,\alpha}(\Gamma_b))} < 1$ by choosing σ big or $\| \zeta \|_{C^{3,\alpha}} \leq \rho$ small enough. Let us define

$$A \stackrel{\text{def}}{=} \left(I + \frac{\lambda}{\sigma} T_{\Gamma}(\zeta) \mathcal{M}_2(\zeta) \right)^{-1}. \quad (4.39)$$

Applying then A to (4.38) leads to

$$\zeta = \frac{\lambda}{\sigma} A\left(\frac{\lambda}{\sigma}, \zeta\right) T_{\Gamma}(\zeta) (G(\lambda, u, \zeta)). \quad (4.40)$$

Since (4.37) and (4.40) together are an equivalent formulation for (4.32) and (4.33), we conclude that (u, ζ) is a solution of the free boundary problem, if (u, ζ) is a fixed point of the mapping

$$\begin{pmatrix} v \\ \xi \end{pmatrix} \mapsto \lambda \begin{pmatrix} T_{\Omega_0}(\xi, F(\lambda, v, \xi), \gamma(\xi)) \\ \sigma^{-1} A\left(\frac{\lambda}{\sigma}, \xi\right) T_{\Gamma}(\xi) (G(\lambda, v, \xi)) \end{pmatrix}. \quad (4.41)$$

This result can be obtained however only, if the right hand sides of (4.37) and (4.40) are well-defined. We need to make sure that $F(\lambda, u, \zeta)$ and $G(\lambda, u, \zeta)$ are elements of the correct spaces.

Lemma 4.14

Let $u \in (H^2(\Omega_0))^3 \cap J(\Omega_0)$ and $\zeta \in V_{\Gamma, C^{3,\alpha}}$, then each summand of $F(\lambda, u, \zeta)$ is an element of a space which is compactly imbedded in $(L^p(\Omega_0))^3$, $2 \leq p < 6$, and $\gamma \in (W^{1,p}(\Omega_0))^3$, $p \geq 2$.

Proof:

The highest derivative of ζ in f is of order three and therefore $f(\lambda, \zeta) \in (C^{0,\alpha}(\Omega_0))^3$. Moreover the imbeddings $(C^{0,\alpha}(\Omega_0))^3 \hookrightarrow (L^p(\Omega_0))^3$ are compact for all positive p .

The term $L_0(\lambda, \zeta)u$ depends on ζ via u_0^0 , which is a part of the basic solution u_0 . The highest derivative of ζ in this term is of order two. Because L_0 is a differential operator of first order with respect to u , the least regular term in $L_0(\lambda, \zeta)$ is of the form $\zeta_{xy} \frac{\partial u}{\partial x}$. This term is a multiplication of a C^1 -function with a H^1 -function. The product and therefore the whole term $L_0(\lambda, \zeta)u$ is in $(H^1(\Omega_0))^3$.

The nonlinear term $Bu = \sum_k u^k \partial_k u$ is the product of an H^1 -function with an H^2 -function. We can estimate Bu using the imbedding $H^1 \hookrightarrow L^6$ in $\mathbf{R}^3$ as follows

$$\begin{aligned} \int_{\Omega_0} \left(\sum_{k=1}^3 u^k \frac{\partial u}{\partial x^k} \right)^2 dx &\leq \text{const.} \sum_{k=1}^3 \int_{\Omega_0} \left(u^k \frac{\partial u}{\partial x^k} \right)^2 dx \\ &\leq \text{const.} \sum_{k=1}^3 \left(\int_{\Omega_0} (u^k)^4 dx \right)^{1/2} \cdot \left(\int_{\Omega_0} \left(\frac{\partial u}{\partial x^k} \right)^4 dx \right)^{1/2} < \infty. \end{aligned}$$

Hence $Bu \in (L^2(\Omega_0))^3$.

We claim that also the first derivatives of Bu are L^2 -functions. For this we need to show that if $u \in H^2$ and $v \in H^1$, then $\frac{\partial}{\partial x}(uv) \in L^2$. We compute that

$$\begin{aligned} \int_{\Omega_0} \left(\frac{\partial}{\partial x}(uv) \right)^2 dx &= \int_{\Omega_0} \left(\frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} \right)^2 dx \\ &\leq \text{const.} \left\{ \int_{\Omega_0} \left(\frac{\partial u}{\partial x} v \right)^2 dx + \int_{\Omega_0} \left(u \frac{\partial v}{\partial x} \right)^2 dx \right\}. \end{aligned}$$

The first integral is finite, since $\frac{\partial u}{\partial x}$ and v are both elements of $H^1 \hookrightarrow L^6$.

The second integral is finite, since $u \in H^2 \hookrightarrow C^0$ for $\Omega_0 \subset \mathbb{R}^3$ and $\frac{\partial v}{\partial x} \in L^2$. Hence we can conclude that

$$F(\lambda, u, \zeta) \in (L^p(\Omega_0))^3, \quad 2 \leq p < 6,$$

by using again the Sobolev imbedding $H^1 \hookrightarrow L^p$. ■

We turn now to $G(\lambda, u, \zeta)$. We observe that for $u \in (H^2(\Omega_0))^3$ and $p \in H^1(\Omega_0)$, $n(\zeta) \cdot t(u) \cdot n(\zeta)|_{\Gamma}$ is not an element of $C^{1,2\alpha}(\Gamma_b)$. To overcome this problem we will approximate u and p by functions w and q with enough regularity.

Take

$$u \in (H^2(\Omega_0))^3 \cap J(\Omega_0), \quad \zeta \in V_{\Gamma, C^3, \alpha}$$

and map

$$(u, \zeta) \mapsto v_1 \stackrel{\text{def}}{=} \lambda T_{\Omega_0}(\zeta, F(\lambda, u, \zeta), \gamma(\zeta)). \quad (4.42)$$

We have seen before that $F(\lambda, u, \zeta) \in (L^6(\Omega_0))^3$ and $\gamma \in (W^{1,6}(\Omega_0))^3$. By the regularity Theorem 3.11 v_1 is an element of $(W^{2,6}(\Omega_0))^3$. Now we do the same procedure once more but with (v_1, ζ) , i.e. :

$$(v_1, \zeta) \mapsto v_2 \stackrel{\text{def}}{=} \lambda T_{\Omega_0}(\zeta, F(\lambda, v_1, \zeta), \gamma(\zeta)). \quad (4.43)$$

Using Sobolev imbeddings one can show that

$$F(\lambda, v_1, \zeta) \in (L^p(\Omega_0))^3, \quad \forall p \geq 2.$$

The critical term in $F(\lambda, v_1, \zeta)$ is $Bv_1 \in (W^{1,6}(\Omega_0))^3$. The Sobolev imbedding

$$(W^{1,6}(\Omega_0))^3 \hookrightarrow (L^p(\Omega))^3, \quad \forall p \geq 2,$$

and the regularity Theorem 3.11 imply that

$$v_2 \in (W^{2,p}(\Omega_0))^3, \quad \forall p \geq 2.$$

Applying the Sobolev imbedding

$$(W^{2,p}(\Omega))^3 \hookrightarrow (C^{1,\tilde{\alpha}}(\Omega_0))^3, \quad 0 \leq \tilde{\alpha} < 1,$$

to v_2 we can conclude that

$$v_2 \in (C^{1,\tilde{\alpha}}(\Omega_0))^3, \quad 0 \leq \tilde{\alpha} < 1.$$

Taking now (v_2, ζ) and doing the procedure another time we obtain:

$$(v_2, \zeta) \mapsto w \stackrel{\text{def}}{=} \lambda T_{\Omega_0}(\zeta, F(\lambda, v_2, \zeta), \gamma(\zeta)). \quad (4.44)$$

It can be shown that $w \in (C^{2,\alpha}(\bar{\Omega}_0))^3$. This time the critical terms are:

(i) the term in $F(\lambda, v_2, \zeta)$ with the third derivative of ζ .

Hence $F(\lambda, v_2, \zeta) \in (C^{0,\alpha}(\Omega_0))^3$.

(ii) $\gamma(\zeta) \in (C^{1,\alpha}(\Gamma_b))^3$.

We remark that by iterating this procedure once more we shall not gain any more regularity. The smoothness of ζ puts an end to this regularization. Here $w \in (C^{2,\alpha}(\bar{\Omega}_0))^3$ corresponds with a pressure $q \in C^{1,\alpha}(\bar{\Omega}_0)$. The pressure q is determined uniquely up to a constant. We choose the constant in q in such a way that

$$\int_{\Gamma_0} n(\zeta) \cdot t(w, q) \cdot n(\zeta) ds = 0.$$

Take now (w, q) and substitute it into $G(\lambda, \cdot, \zeta)$. i.e.

$$G(\lambda, w, \zeta) = \beta(\lambda, \zeta) + n(\zeta) \cdot t(w, q) \cdot n(\zeta) |_{\Gamma}. \quad (4.45)$$

We observe that the term in $G(\lambda, w, \zeta)$ which restricts the regularity most is $t(w, q) \in C^{1,\alpha}(\Gamma_b)$. This implies that

$$\xi \stackrel{\text{def}}{=} \frac{\lambda}{\sigma} A\left(\frac{\lambda}{\sigma}, \zeta\right) T_{\Gamma}(\zeta) G(\lambda, w, \zeta) \in C^{3,\alpha}(\Gamma_b).$$

Remark 4.15

By the procedure described above we map $(u, \zeta) \in (H^2(\Omega_0))^3 \times C^{3,\alpha}(\Gamma_b)$ into $(v, \xi) \stackrel{\text{def}}{=} \left(\lambda T_{\Omega_0}(\zeta, F(\lambda, u, \zeta), \gamma(\zeta)); \frac{\lambda}{\sigma} A T_{\Gamma}(G(\lambda, w, \zeta)) \right)$,

and (v, ξ) is again an element in $(H^2(\Omega_0))^3 \times C^{3,\alpha}(\Gamma_b)$. Up to this point we have done only regularity considerations. But in order to set up an iteration scheme for a fixed point equation, we need to estimate $G(\lambda, w, \zeta)$, since there are no fixed points for (4.29), if $\|G(\lambda, u, \zeta)\|_{C^{1,\alpha}}$ is not small enough.

Let us summarize the results from the construction of the fixed point equation in the following way:

Lemma 4.16

- (i) The fixed points of the mapping (4.41) are elements of $(C^{2,\alpha}(\Omega_0))^3 \times C^{3,\alpha}(\Gamma_b)$.
- (ii) The mapping

$$\begin{pmatrix} u \\ \zeta \end{pmatrix} \mapsto \begin{pmatrix} \lambda T_{\Omega_0}(\zeta, F(\lambda, u, \zeta), \gamma(\zeta)) \\ \lambda \sigma^{-1} A(\frac{\lambda}{\sigma}, \zeta) T_{\Gamma}(\zeta)(G(\lambda, w, \zeta)) \end{pmatrix} \stackrel{\text{def}}{=} \lambda T(u, \zeta) \quad (4.46)$$

from $(H^2(\Omega_0))^3 \times C^{3,\alpha}(\Gamma_b)$ into itself is not compact.

Proof:

- (i) In case (u, ζ) is a fixed point of (4.41), the construction of w is simply a bootstrapping procedure for u .
- (ii) The regularity of w can not be raised further, since the process is bounded by the regularity of ζ . ■

We still need to estimate $G(\lambda, w, \zeta)$.

Lemma 4.17

Let $u \in B_r(0) \subset (H^2(\Omega_0))^3$ and $\zeta \in B_\rho(0) \subset C^{3,\alpha}(\Gamma_b)$. Furthermore assume that $\lambda \in [0, \Lambda_0]$. If $r > 0$, $\rho > 0$ are small enough and σ is big enough, then

$$\frac{\lambda}{\sigma} \|G(\lambda, w, \zeta)\|_{C^{1,\alpha}} < b_0^{-1},$$

where b_0^{-1} is the constant in Lemma 4.13.

Proof:

From Theorem 3.11 and the construction of w we obtain that

$$\|w\|_{C^{2,\alpha}(\bar{\Omega}_0)} + \|p\|_{C^{1,\alpha}(\bar{\Omega}_0)} \leq c_1\rho + c_2r,$$

where c_1 and c_2 are positive constants. This implies that

$$\|G(\lambda, w, \zeta)\|_{C^{1,\alpha}} \leq c_3\rho + c_4r. \quad (4.47)$$

From the proof of Lemma 4.13 we read off that the constant b_0 is of the form $b_0 = c_5^{-1}(\rho^{-1} + c_6)$, which we estimate by

$$b_0^{-1} = \frac{c_5}{\rho^{-1} + c_6} = \frac{\rho c_5}{1 + \rho c_6} \geq c_5\rho(1 - c_6\rho). \quad (4.48)$$

Here we have chosen ρ so small that $c_6\rho < 1$. Now take σ so large that $c_5 = \frac{\Lambda_0}{\sigma}c_3 + d$, with some $d > 0$. Then we can estimate

$$\begin{aligned} \frac{\Lambda_0}{\sigma} \|G(\lambda, w, \zeta)\|_{C^{1,\alpha}} &\leq \frac{\Lambda_0}{\sigma} c_3\rho + \frac{\Lambda_0}{\sigma} c_4r \\ &= c_5\rho - d\rho + \frac{\Lambda_0}{\sigma} c_4r \\ &= (c_5\rho - c_5c_6\rho^2) + (c_5c_6\rho^2 + \frac{\Lambda_0}{\sigma} c_4r - d\rho). \end{aligned}$$

Now we might choose σ sufficiently large and ρ and r sufficiently small such that $c_5c_6\rho^2 + \frac{\Lambda_0}{\sigma} c_4r - d\rho \leq 0$. Using this inequality in our estimate above we finally obtain:

$$\frac{\Lambda_0}{\sigma} \|G(\lambda, w, \zeta)\|_{C^{1,\alpha}} \leq c_5\rho(1 - c_6\rho) \leq b_0^{-1}. \quad \blacksquare$$

Remark 4.18

At this point let us summarize what conditions the radius ρ has to satisfy:

(i) We need to choose $\rho > 0$ small to get a unique solution of

$$\mathcal{M}_1(\xi)(\xi)\zeta = g, \quad \forall \xi \in B_\rho(0) \subset V_{\Gamma, C^3, \alpha}.$$

Let us assume that $\rho = \rho_0$ is small enough.

(ii) Choose $\sigma > 0$ so large that

$$c_5 - \frac{\Lambda_0}{\sigma} c_3 \stackrel{\text{def}}{=} d > 0.$$

We may assume that σ_0 is large enough.

(iii) Now choose $\rho \leq \rho_0$, $\sigma \geq \sigma_0$, if necessary, and $r > 0$ small enough such that

$$c_5 c_6 \rho^2 + \frac{\Lambda_0}{\sigma} c_4 r - d\rho < 0.$$

When we have chosen ρ, σ and r such that (i)–(iii) is satisfied, the proof of the last theorem shows that then

$$\|G(\lambda, w, \zeta)\|_{C^{1,\alpha}} \leq b_0^{-1}$$

and hence satisfies the smallness requirement to obtain a solution $\zeta \in B_{\rho(0)}$.

Now we summarize the results we have obtained so far in

Theorem 4.19

If

$$(u, \zeta) \in B_r(0) \times B_\rho(0) \subset (H^2(\Omega_0))^3 \times C^{3,\alpha}(\Gamma_b),$$

then (i) and (ii) below are equivalent.

(i) (u, ζ) is a solution of (4.32) and (4.33).

(ii) (u, ζ) is a fixed point of (4.46). ■

We shall now take a look at the linearization of $T(u, \zeta)$, which we call K . For the very same reason that T is not a compact operator, K is not compact either.

Definition 4.20

Let us call the linearizations of F, γ and G , respectively, L_F, L_γ and L_G . Then

$$K(u, \zeta) \stackrel{\text{def}}{=} \left(T_{\Omega_0}(\zeta, L_F, L_\gamma); \sigma^{-1} A T_\Gamma(L_G) \right)$$

is the linearization of T .

The linear function L_G is a sum of two functions, $L_G = L_{\beta_1} + L_{t(w)}$, where $L_{\beta_1}(\lambda, \zeta) \in C^{2,\alpha}(\Gamma_b)$. Hence

$$K_{\beta_1}(\zeta) \stackrel{\text{def}}{=} \sigma^{-1} A T_\Gamma(L_{\beta_1}(\lambda, \zeta)) \in \mathcal{L}(C^{3,\alpha}(\Gamma_b))$$

is a compact operator. Furthermore let us define

$$R(u, \zeta) \stackrel{\text{def}}{=} \sigma^{-1} A T_{\Gamma}(L_{t(w)}(u, \zeta)). \quad (4.49)$$

We might write K then in the form

$$K(u, \zeta) = (T_{\Omega_0}(\zeta, L_F, L_{\gamma}); K_{\beta_1} + R(u, \zeta)). \quad (4.50)$$

Theorem 4.21

If σ is large enough, then $I - K$ is a Fredholm operator with index $\text{ind}(I - K) = 0$.

Proof:

Let us write $I - K = I - K_0 + \tilde{R}$, where

$$K_0(u, \zeta) = (T_{\Omega_0}(\zeta, L_F, L_{\gamma}); K_{\beta_1}(\zeta))$$

and

$$\tilde{R}(u, \zeta) = (0; R(u, \zeta)).$$

The operator K_0 is compact. Hence by the equation above $I - K$ is a perturbation of a Fredholm operator of index zero.

Claim 4.22

If $\|R\|$ is small, then $I - K$ is a Fredholm operator of index zero.

Proof:

(i)

The set of all Fredholm operators is open in the set of the bounded linear operators. Therefore if F is a Fredholm operator and S is a linear bounded operator with $\|S\|$ smaller than some given positive number, then $F + S$ is also a Fredholm operator (see [Satz 25.2][Hir/Schar]).

(ii)

Let Φ be the set of all Fredholm operators. The map $\text{ind} : \Phi \rightarrow \mathbb{Z}$ is continuous on each connected component of Φ (see [Hir/Schar, ibid]). If $\|R\|$ is so small that $I - K_0 + R$ and $I - K_0$ are in the same connected component, then the mapping $t \mapsto I - K_0 + tR$ is continuous on $[0, 1]$. Hence $\text{ind}(I - K_0 + R) = \text{ind}(I - K_0)$.

It remains to show that $\|R\|$ is small enough, if σ is sufficiently large. We can estimate:

$$\|R\| \leq \sigma^{-1} \|A(\frac{\lambda}{\sigma})\| \cdot \|T_{\Gamma}\| \cdot \|L_{t(w)}\|, \quad (4.51)$$

and

$$\|A(\frac{\lambda}{\sigma})\| = \left\| \left(I + \frac{\lambda}{\sigma} T_{\Gamma} \mathcal{M}_2(\xi) \right)^{-1} \right\| \leq 1 + \Lambda_0 \|T_{\Gamma} \mathcal{M}_2(\xi)\| \stackrel{def}{=} c_0,$$

since we can assume that $\sigma \geq 1$. It is clear from (4.51) that $\|R\|$ is small if σ is large enough. ■

The next step now would be to analyze the spectrum of the linearization K . Since $I - K$ is a Fredholm operator of index zero, $\lambda = \lambda_0$ is a bifurcation point of $I - T$, if λ_0 is an eigenvalue of odd multiplicity of K . This is a well-known result in bifurcation theory (see [Dei, p.383]). Suppose that $u \in X_u$ is a solution, $\lambda \in \mathbf{R}$ is the bifurcation parameter, $X \stackrel{def}{=} X_u \times \mathbf{R}$ is a Banach space and, moreover, $I - K$ is a compact operator. Then the continuum of nontrivial solutions branching off from λ_0 is either unbounded in the Banach space X or connected with another eigenvalue of odd multiplicity. It happens quite often that the first eigenvalue is of odd multiplicity or even simple. If this is the case and the nontrivial branch, which starts from the first eigenvalue, turns to smaller values of λ and if there exists a lower bound for the values of λ , the solutions u have to go to ∞ in the norm of $\|\cdot\|_{X_u}$. Another situation which might occur is that there exists only one eigenvalue of odd multiplicity and that the solutions u are a priori bounded in $\|\cdot\|_{X_u}$. Then the nontrivial branch has to go to ∞ in λ , i.e., for all $\lambda > \lambda_0$ there exists a nontrivial solution. The numerical computations in the next chapter will show that a branch of nontrivial solutions starts from the first eigenvalue (see Fig. 5.1 and 5.2).

Chapter 5

Numerical computations for the two- dimensional flow

In this chapter we describe how the two dimensional problem can be rewritten as a system of ordinary differential equations. For this system numerical computations have been carried out with the program AUTO [Doe/Ker]. AUTO is a computer program which is designed for solving bifurcation problems of algebraic and ordinary differential equations. It is applicable in cases where a basic solution is known for a certain value of the bifurcation parameter. Starting from this solution AUTO computes a one dimensional solution manifold using the pseudo- arclength method. When the solution space becomes two- dimensional, AUTO labels this point as a bifurcation point. The user can restart from this labeled point with a prescribed starting direction which is different from the direction of the already computed solution branch. The direction is taken in the (parameter, L^2 -norm of the solution)- plane. Thus a new solution branch can be discovered and computed.

The reduction of the free boundary value problem to two dimensions is necessary for transforming the problem into a system of ordinary differential equations and thus being able to use AUTO. Moreover one expects anyway that the first bifurcation might occur in the x - and z - components of the velocity. There are only forces in those directions acting on the fluid.

For a fluid in two dimensions which is incompressible, we can introduce the stream function $\psi(x, z)$. Writing everything in terms of the stream function we obtain a boundary value problem of fourth order. Then we expand

the stream function into a Fourier series with respect to x . The boundary value problem now becomes an infinite system of ordinary differential equations for the Fourier coefficients $\psi_n(z)$. It is reasonable to cut off this system and consider only the first two or three coefficients for example. This procedure is justified, since the first modes are connected with the largest wave lengths. Besides this it is also necessary to take only very few terms to keep the CPU- time in tolerable bounds.

5.1 The BVP for the stream function

A two dimensional incompressible fluid flow can be described by a stream function, i.e., there exists a function $\psi(x, z)$ such that

$$\begin{aligned} u^1 &= \psi_z, \\ u^3 &= -\psi_x. \end{aligned} \quad (5.1)$$

The existence of the stream function is guaranteed by Poincaré's lemma (see e.g. [Bot/Tu]) and $\operatorname{div} u = u_x^1 + u_z^3 = 0$.

It is helpful to rewrite the differential equation first as a vorticity equation, before writing it in terms of the stream function. The vorticity ξ is defined by

$$\xi = u_x^3 - u_z^1, \quad (5.2)$$

which implies that

$$\xi = -\Delta\psi. \quad (5.3)$$

The Navier- Stokes equation written as vorticity equation is

$$-\frac{1}{\lambda}\Delta\xi + u_k\partial_k\xi_0 + u_{0k}\partial_k\xi + u_k\partial_k\xi = \partial_1f^3 - \partial_3f^1. \quad (5.4)$$

Here we have used the notation $x^1 = x$ and $x^3 = z$ and the summation convention for convenience. Furthermore u_0 is the basic solution and ξ_0 the respective basic vorticity. We combine now (5.3) and (5.4) and obtain a fourth order differential equation,

$$\begin{aligned} -\frac{1}{\lambda}\Delta^2\psi + \psi_z\xi_{0,x} - \psi_x\xi_{0,z} - u_0^1\Delta\psi_x - u_0^3\Delta\psi_z \\ -\psi_z\Delta\psi_x + -\psi_x\Delta\psi_z = f_x^3 - f_z^1. \end{aligned} \quad (5.5)$$

Now we shall write down the boundary conditions in terms of the stream function.

Periodic boundary conditions

$$\partial^\alpha \psi(0, z) = \partial^\alpha \psi(x_0, z), \quad 1 \leq |\alpha| \leq 2. \quad (5.6)$$

Non slip condition at Γ_b

$$\begin{aligned} \psi_x(x, 0) &= 0 \\ \psi_z(x, 0) &= -\frac{\mu}{\lambda} \frac{\zeta}{1 + \zeta}. \end{aligned} \quad (5.7)$$

Here we have called the parameter μ_1 from the chapters before simply μ .

Velocity is tangential at Γ

The normal unit vector n at Γ is given by (see 1.27)

$$n = \frac{1}{\sqrt{1 + \zeta'^2}} \begin{pmatrix} -\zeta' \\ 1 \end{pmatrix}. \quad (5.8)$$

Hence $u \cdot n|_\Gamma = 0$ becomes

$$\psi_z(x, 1 + \zeta)\zeta' + \psi(x, 1 + \zeta) = 0. \quad (5.9)$$

Tangential stress condition

The unit tangential vector at Γ is (see (1.32))

$$\tau = \frac{1}{\sqrt{1 + \zeta'^2}} \begin{pmatrix} 1 \\ \zeta' \end{pmatrix}. \quad (5.10)$$

Furthermore we need to compute the stress tensor:

$$t(u) = \begin{pmatrix} -p + 2\psi_{zz} & \psi_{zz} - \psi_{xx} \\ \psi_{zz} - \psi_{xx} & -p + 2\psi_{xx} \end{pmatrix}. \quad (5.11)$$

Combining (5.8), (5.10) and (5.11) with the equation for γ_1 (see (3.13)) we obtain the tangential stress condition

$$\begin{aligned} (\psi_{zz}(x, 1 + \zeta) - \psi_{xx}(x, 1 + \zeta))(1 - \zeta'^2)(1 + \zeta)^{1/2} + \\ 2(1 - \frac{\mu}{\lambda})\zeta''(1 + \zeta'^2)^{1/2} = 0. \end{aligned} \quad (5.12)$$

Normal stress condition

For the normal stress condition we combine the equation for unit normal and the stress tensor with the equation for β (see (1.2)), which leads to

$$\begin{aligned} & \frac{\sigma}{\lambda} \zeta'' (1 + \zeta)^2 + \\ & 2(1 - \frac{\mu}{\lambda}) \zeta' \left\{ (1 + \zeta'^2)^{3/2} - \zeta'' (1 + \zeta) (1 + \zeta'^2)^{1/2} \right\} + \\ & \{ p(x, 1 + \zeta) - 2\psi_{xz}(x, 1 + \zeta) \} (1 + \zeta)^2 (1 + \zeta'^2)^{3/2} + \\ & 2 \{ \psi_{zz}(x, 1 + \zeta) - \psi_{xx}(x, 1 + \zeta) \} \zeta' (1 + \zeta)^2 (1 + \zeta'^2)^{1/2} = 0. \end{aligned} \quad (5.13)$$

5.2 Rescaling

In this section we shall carry out a rescaling such that we can consider the problem on the unit square:

$$\begin{aligned} \bar{x} &= \frac{x}{x_0} \stackrel{\text{def}}{=} \alpha x, \\ \bar{z} &= \frac{z}{1 + \zeta}. \end{aligned} \quad (5.14)$$

Hence the partial derivatives of any function $h(x, z)$ are

$$\frac{\partial h}{\partial \bar{x}} = \alpha \frac{\partial h}{\partial x}$$

and

$$\frac{\partial h}{\partial \bar{z}} = \frac{1}{1 + \zeta} \frac{\partial h}{\partial z}.$$

Then the equations of the rescaled problem are:

$$\begin{aligned}
& \frac{1}{\lambda} \left\{ \alpha^4 \frac{\partial^4 \psi}{\partial \bar{x}} + \frac{1}{(1+\zeta)^4} + 2 \frac{\alpha^2}{(1+\zeta)^2} \frac{\partial^4 f}{\partial \bar{x}^2 \partial \bar{z}^2} \right\} + \\
& \quad \frac{\alpha}{1+\zeta} \left(\frac{\partial \psi}{\partial \bar{z}} \frac{\partial \xi}{\partial \bar{x}} - \frac{\partial \psi}{\partial \bar{x}} \frac{\partial \xi}{\partial \bar{z}} \right) - \\
& \quad u_0^1 \left(\alpha^3 \frac{\partial^3 \psi}{\partial \bar{x}^3} + \frac{\alpha}{(1+\zeta)^2} \frac{\partial^3 \psi}{\partial \bar{x} \partial \bar{z}^2} \right) - \\
& \quad \frac{u_0^3}{1+\zeta} \left(\alpha^2 \frac{\partial^3 \psi}{\partial \bar{x}^2 \partial \bar{z}} + \frac{1}{(1+\zeta)^2} \frac{\partial^3 \psi}{\partial \bar{z}^3} \right) - \\
& \quad \frac{1}{1+\zeta} \frac{\partial \psi}{\partial \bar{z}} \left(\alpha^3 \frac{\partial^3 \psi}{\partial \bar{x}^3} + \frac{\alpha}{(1+\zeta)^2} \frac{\partial^3 \psi}{\partial \bar{x} \partial \bar{z}^2} \right) - \\
& \quad \frac{1}{1+\zeta} \frac{\partial \psi}{\partial \bar{x}} \left(\alpha^3 \frac{\partial^3 \psi}{\partial \bar{x}^2 \partial \bar{z}} + \frac{\alpha}{(1+\zeta)^2} \frac{\partial^3 \psi}{\partial \bar{z}^3} \right) = \alpha \frac{\partial f^3}{\partial \bar{x}} - \frac{1}{1+\zeta} \frac{\partial f^1}{\partial \bar{z}},
\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
\xi_0 &= \alpha \frac{\partial u_0^3}{\partial \bar{x}} - \frac{1}{1+\zeta} \frac{\partial u_0^1}{\partial \bar{z}}, \\
f^1 &= \frac{1}{\lambda} \left(\alpha^2 \frac{\partial^2 u_0^1}{\partial \bar{x}^2} + \frac{1}{(1+\zeta)^2} \frac{\partial^2 u_0^1}{\partial \bar{z}^2} \right) - \alpha u_0^1 \frac{\partial u_0^1}{\partial \bar{x}} - \frac{u_0^3}{1+\zeta} \frac{\partial u_0^1}{\partial \bar{z}} - \alpha \frac{\partial p_0}{\partial \bar{x}}, \\
f^3 &= -\frac{\theta}{\lambda} + \frac{1}{\lambda} \left(\alpha^2 \frac{\partial^2 u_0^3}{\partial \bar{x}^2} + \frac{1}{(1+\zeta)^2} \frac{\partial^2 u_0^3}{\partial \bar{z}^2} \right) - \\
& \quad \alpha u_0^1 \frac{\partial u_0^3}{\partial \bar{x}} - \frac{u_0^3}{1+\zeta} \frac{\partial u_0^3}{\partial \bar{z}} - \alpha \frac{\partial p_0}{\partial \bar{z}}
\end{aligned}$$

and

$$u_0 = \frac{1}{1+\zeta} \left(\begin{array}{c} 2\bar{z} - \bar{z}^2 - \frac{\mu}{\lambda} \\ \alpha \zeta' (2\bar{z}^2 - \bar{z}^3 - \frac{\mu}{\lambda}) \end{array} \right).$$

For the rescaled boundary equations we obtain:

Non slip condition at Γ_b

$$\begin{aligned}\frac{\partial \psi}{\partial \bar{x}}(\bar{x}, 0) &= 0 \\ \frac{\partial \psi}{\partial \bar{z}}(\bar{x}, 0) + \frac{\mu}{\lambda} \zeta &= 0.\end{aligned}\tag{5.16}$$

Velocity is tangential at Γ

$$\frac{\partial \psi}{\partial \bar{z}}(\bar{x}, 1) \zeta' + (1 + \zeta) \frac{\partial \psi}{\partial \bar{x}}(\bar{x}, 1) = 0.\tag{5.17}$$

Tangential stress condition

$$\begin{aligned}\left(\frac{\partial^2 \psi}{\partial \bar{z}^2}(\bar{x}, 1) - \alpha^2 (1 + \zeta)^2 \frac{\partial^2 \psi}{\partial \bar{x}^2}(\bar{x}, 1) \right) (1 - \alpha^2 \zeta'^2) + \\ 2\alpha^2 \left(1 - \frac{\mu}{\lambda} \right) \zeta'' (1 + \zeta)^{3/2} (1 + \alpha^2 \zeta'^2)^{1/2} = 0.\end{aligned}\tag{5.18}$$

Normal stress condition

$$\begin{aligned}\frac{\sigma}{\lambda} \alpha^2 \zeta'' (1 + \zeta)^2 + \\ 2 \left(1 - \frac{\mu}{\lambda} \right) \alpha \zeta' \left\{ (1 + \alpha^2 \zeta'^2)^{3/2} - \alpha^2 \zeta'' (1 + \zeta) (1 + \alpha^2 \zeta'^2)^{1/2} \right\} + \\ \left\{ p(\bar{x}, 1) (1 + \zeta) - 2\alpha \frac{\partial^2 \psi}{\partial \bar{x} \partial \bar{z}}(\bar{x}, 1) \right\} (1 + \zeta) (1 + \alpha^2 \zeta'^2)^{3/2} + \\ 2 \left\{ \frac{\partial^2 \psi}{\partial \bar{z}^2}(\bar{x}, 1) - \alpha^2 (1 + \zeta)^2 \frac{\partial^2 \psi}{\partial \bar{x}^2}(\bar{x}, 1) \right\} \alpha \zeta' (1 + \alpha^2 \zeta'^2)^{1/2} = 0.\end{aligned}\tag{5.19}$$

5.3 Fourier transformation

In order to obtain a system of ordinary differential equations we expand $\psi(x, z)$, $p(x, z)$ and $\zeta(x)$ into a Fourier series with respect to x , i.e. ,

$$\psi(x, z) = \sum_{n=-\infty}^{+\infty} \psi_n(z) e^{2n\pi i x}, \quad (5.20)$$

$$p(x, z) = \sum_{n=-\infty}^{+\infty} p_n(z) e^{2n\pi i x}, \quad (5.21)$$

$$\zeta(x) = \sum_{n=-\infty}^{+\infty} \zeta_n e^{2n\pi i x}, \quad (5.22)$$

with

$$\left. \begin{aligned} \psi_n(z) &= \bar{\psi}_{-n}(z), \\ p_n(z) &= \bar{p}_{-n}(z), \\ \zeta_n &= \bar{\zeta}_{-n}. \end{aligned} \right\} \quad (5.23)$$

We observe that the pressure is contained only in the normal stress condition (5.19). We can obtain the Fourier coefficients for the pressure by solving the Navier - Stokes equations for $\text{grad } p$ and evaluating it at $z = 1$. We want to call $\bar{x}$, $\bar{z}$ in this section x , z .

The differential equations for $\psi_n(z)$

If g is any function, then the k^{th} - Fourier coefficient of g and $(1 + g)^2 g'$ are written as g_k and $((1 + g)^2 g')_k$, respectively. Using this notation the

differential equations for $\psi_n(z)$ are the following:

$$\begin{aligned}
\psi_n^{(4)}(z) = & -(2\pi i\alpha)^4 \sum_{k+l=n} \left((1+\zeta)^4\right)_k l^4 \psi_l - \\
& 2(2\pi i\alpha)^2 \sum_{k+l=n} \left((1+\zeta)^2\right)_k l^2 \psi_l'' - \\
& \lambda(2\pi i\alpha) \sum_{k+l=n} k \hat{\xi}_k^1 \psi_l' + \lambda(2\pi i\alpha) \sum_{k+l=n} \hat{\xi}_k^2 l \psi_l + \\
& \left(\lambda(2z - z^2) - \mu \right) \left[(2\pi i\alpha)^3 \sum_{k+l=n} \left((1+\zeta)^3\right)_k l^3 \psi_l + \right. \\
& \quad \left. (2\pi i\alpha) \sum_{k+l=n} (1+\zeta)_k l \psi_l'' \right] + \\
& \alpha \left(\lambda(2z^2 - z^3) - \mu z \right) \left[(2\pi i\alpha)^2 \sum_{k+l=n} \left(\zeta'(1+\zeta)^2\right)_k l^2 \psi_l' + \right. \\
& \quad \left. \sum_{k+l=n} (\zeta')_k \psi_l''' \right] + \\
& \lambda(2\pi i\alpha)^3 \sum_{j+k+l=n} \left((1+\zeta)^3\right)_j \psi_k' l^3 \psi_l + \\
& \lambda(2\pi i\alpha) \sum_{j+k+l=n} (1+\zeta)_j \psi_k' l \psi_l'' + \\
& \lambda(2\pi i\alpha)^3 \sum_{j+k+l=n} \left((1+\zeta)^3\right)_j k \psi_k l^2 \psi_l' + \\
& \lambda(2\pi i\alpha) \sum_{j+k+l=n} (1+\zeta)_j k \psi_k \psi_l''' + \\
& \lambda \left((1+\zeta)^3 \frac{\partial f^1}{\partial z} \right)_n - \lambda \alpha \left((1+\zeta)^4 \frac{\partial f^3}{\partial x} \right)_n .
\end{aligned} \tag{5.24}$$

Doing some tedious computations one obtains:

$$\begin{aligned}\hat{\xi}^1 &\stackrel{def}{=} (1+\zeta)^3 \frac{\partial \xi_0}{\partial x} \\ &= \alpha^2(2z^2 - z^3 - \frac{\mu}{\lambda}z) \left(\zeta'''(1+\zeta)^2 - 3\zeta'\zeta''(1+\zeta) + 2\zeta'^3 \right) \\ &\quad + (4 - 4z)\zeta',\end{aligned}\quad (5.25)$$

$$\begin{aligned}\hat{\xi}^2 &\stackrel{def}{=} (1+\zeta)^3 \frac{\partial \xi_0}{\partial z} \\ &= \alpha^2(4z - 3z^2 - \frac{\mu}{\lambda}) \left(\zeta''(1+\zeta)^2 - \zeta'^2(1+\zeta) \right) + 2(1+\zeta),\end{aligned}\quad (5.26)$$

$$\begin{aligned}(1+\zeta)^3 \frac{\partial f^1}{\partial x} &= \frac{\alpha^2}{\lambda}(2-2z)(\zeta''(1+\zeta) + 2\zeta'^2) \\ &\quad + 2\alpha(3z^2 - 2z^3 - \frac{\mu}{\lambda})\zeta'\end{aligned}\quad (5.27)$$

and

$$\begin{aligned}(1+\zeta)^4 \frac{\partial f^3}{\partial x} &= \frac{\alpha^3}{\lambda}(2z^2 - z^3 - \frac{\mu}{\lambda}) \left(\zeta^{(4)}(1+\zeta)^3 - 4\zeta'\zeta'''(1+\zeta)^2 \right. \\ &\quad \left. 3\zeta''^2(1+\zeta)^2 + 12\zeta'^2\zeta''(1+\zeta) - 6\zeta'^4 \right) + \\ &\quad \frac{\alpha}{\lambda}(4-6z) \left(\zeta''(1+\zeta) - 3\zeta'^2 \right) - \\ &\quad \alpha^2(2z - z^2 - \frac{\mu}{\lambda})(2z^2 - z^3 - \frac{\mu}{\lambda}z) \\ &\quad \left(\zeta'''(1+\zeta)^2 - 2\zeta'\zeta''(1+\zeta) \right) - \\ &\quad 2\alpha^2(2z^2 - z^3 - \frac{\mu}{\lambda})(z + z^2) \left(2\zeta'\zeta''(1+\zeta) - 3\zeta'^3 \right) - \\ &\quad \frac{\theta}{\lambda}\zeta'(1+\zeta)^2.\end{aligned}\quad (5.28)$$

Now we turn to the boundary conditions.

Non slip condition at Γ_b

$$\begin{aligned}\psi_n(0) &= 0 \\ \psi'_n(0) + \frac{\mu}{\lambda}\zeta_n &= 0\end{aligned}\quad (5.29)$$

Velocity is tangential at Γ

$$\sum_{k+l=n} \psi'_k(1) l \zeta_l + \sum_{k+l=n} (1+\zeta)_k l \psi_l(1) = 0. \quad (5.30)$$

Tangential stress condition

$$\begin{aligned} & \sum_{k+l=n} (1 - \alpha^2 \zeta'^2)_k \psi''_l(1) - \\ & - (2\pi i \alpha)^2 \sum_{k+l=n} \left((1+\zeta)^2 (1 - \alpha^2 \zeta'^2) \right)_k l^2 \psi_l(1) + \\ & + 2\alpha^2 \left(1 - \frac{\mu}{\lambda} \right) \left(\zeta''(1+\zeta)^{3/2} (1 + \alpha^2 \zeta'^2)^{1/2} \right)_n = 0. \end{aligned} \quad (5.31)$$

Normal stress condition

$$\begin{aligned} & \frac{\sigma}{\lambda} \alpha^2 \left(\zeta''(1+\zeta)^2 \right)_n + 2\alpha \left(1 - \frac{\mu}{\lambda} \right) \left\{ \left(\zeta'(1 + \alpha^2 \zeta'^2)^{3/2} \right)_n - \right. \\ & \left. - \left(\zeta' \alpha^2 \zeta''(1+\zeta)(1 + \alpha^2 \zeta'^2)^{1/2} \right)_n \right\} + \\ & + \left(p(x, 1)(1+\zeta)^2 (1 + \alpha^2 \zeta'^2)^{3/2} \right)_n - \\ & - 2(2\pi i \alpha) \sum_{k+l=n} k \psi'_k(1) \left((1+\zeta)(1 + \alpha^2 \zeta'^2)^{3/2} \right)_l + \\ & + 2\alpha \sum_{k+l=n} \psi''_k(1) \left(\zeta'(1 + \alpha^2 \zeta'^2)^{1/2} \right)_l - \\ & - 2\alpha (2\pi i \alpha)^2 \sum_{k+l=n} k^2 \psi_k(1) \left(\zeta'(1+\zeta)^2 (1 + \alpha^2 \zeta'^2)^{1/2} \right)_l = 0. \end{aligned} \quad (5.32)$$

The pressure at the free boundary can be computed by solving the first of the Navier- Stokes equations for the pressure term and evaluate it at the free boundary:

$$\alpha \frac{\partial p}{\partial x} = f^1 - h^1, \quad (5.33)$$

where

$$\begin{aligned}
h^1 = & -\frac{1}{\lambda} \frac{1}{1+\zeta} \left(\alpha^2 \frac{\partial^3 \psi}{\partial x^2 \partial z^2} + \frac{1}{(1+\zeta)^2} \frac{\partial^3 \psi}{\partial z^3} \right) + \\
& \frac{\alpha}{(1+\zeta)^3} \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\alpha}{(1+\zeta)^2} \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial z^2} + \\
& \frac{\alpha u_0^1}{1+\zeta} \frac{\partial^2 \psi}{\partial x \partial z} + \frac{u_0^3}{(1+\zeta)^2} \frac{\partial^2 \psi}{\partial z^2} + \\
& \frac{\alpha}{1+\zeta} \frac{\partial \psi}{\partial z} \frac{\partial u_0^1}{\partial x} - \frac{1}{(1+\zeta)^2} \frac{\partial \psi}{\partial x} \frac{\partial u_0^1}{\partial z}.
\end{aligned} \tag{5.34}$$

Hence the equation for the Fourier coefficients evaluated at the free surface becomes

$$\begin{aligned}
((1+\zeta)^3 h^1)_n(1) = & -\frac{1}{\lambda} (2\pi i \alpha)^2 \\
& \left(\sum_{k+l=n} \left((1+\zeta)^2 \right)_k l^2 \psi'_l(1) + \psi'''_n(1) \right) + \\
& (2\pi i \alpha) \sum_{k+l=n} \psi'_k(1) j \psi'_l(1) - \\
& (2\pi i \alpha) \sum_{j+k+l=n} (1+\zeta)_j \psi_k(1) l \psi'''_l(1) + \\
& (2\pi i \alpha) \left(1 - \frac{\mu}{\lambda} \right) \sum_{k+l=n} (1+\zeta)_k l \psi_l(1) + \\
& \alpha \left(1 - \frac{\mu}{\lambda} \right) \sum_{k+l=n} \zeta'_k \psi''_l(1) - \\
& \alpha \left(1 - \frac{\mu}{\lambda} \right) \sum_{k+l=n} \zeta'_k \psi'_l(1).
\end{aligned} \tag{5.35}$$

Furthermore the computations for f_1 lead to:

$$f^1 = -\frac{\alpha^2}{\lambda}(2z - z^2 - \frac{\mu}{\lambda}) \left(\frac{\zeta''}{(1\zeta)^2} - \frac{2\zeta'^2}{(1+\zeta)^3} \right) + \frac{2}{\lambda} \frac{1}{(1+\zeta)^3} + \alpha \left[(2z - z^2 - \frac{\mu}{\lambda})^2 - 2(1-z)(2z^2 - z^3 - \frac{\mu}{\lambda}) \right] \frac{\zeta'}{(1+\zeta)^3} - \frac{2\alpha}{\lambda}, \quad (5.36)$$

and the equation for the Fourier coefficients evaluated at the free surface is

$$\begin{aligned} ((1+\zeta)^3 f^1)_n(1) &= -\frac{\alpha^2}{\lambda} (1 - \frac{\mu}{\lambda}) \left[(\zeta''(1\zeta))_n - (2\zeta'^2)_n \right] \\ &+ \alpha (1 - \frac{\mu}{\lambda})^2 (\zeta')_n - \frac{2\alpha}{\lambda} (1 - (1+\zeta)^3)_n. \end{aligned} \quad (5.37)$$

We can now obtain the Fourier coefficients for the pressure at the free boundary from the following set of equations:

$$\begin{aligned} \left(\frac{\partial p}{\partial x} (1+\zeta)^3 \right)_n(1) &= (2\pi i \alpha) \sum_{k+l=n} k p_k(1) ((1+\zeta)^3)_l \\ &= ((1+\zeta)^3 f^1)_n - ((1+\zeta)^3 h^1)_n. \end{aligned} \quad (5.38)$$

We observe that the constant term in the Fourier expansion for the pressure can not be determined by those equations above. As a matter of fact we could not expect this, since the pressure is determined by the solution (u, ζ) only up to a constant. The constant term is a free parameter in our problem. On the other hand the constant term in the Fourier expansion of ζ is determined to be zero by the fact that the volume of Ω should not change.

5.4 Numerical results

The actual computations have been carried out with an Fourier expansion up to the first mode, i.e. ,

$$\psi(x, z) = \psi_{-1}(z)e^{-2\pi i x} + \psi_0(z) + \psi_1(z)e^{2\pi i x}, \quad (5.39)$$

etc.. Computations with a Fourier expansion up to the second mode did not seem to provide reliable results. Besides this the CPU- time on a VAX 8600 for the two modes computations has been in the magnitude of 20 - 30

hours. Therefore more accurate computations would have been very costly with a vague perspective for success. For the actual computations we have chosen the parameters to be

$$\begin{aligned}\alpha &= 1.0 \cdot 10^{-2}, \\ \theta &= 9.81 \cdot 10^6, \\ \sigma &= 72.75, \\ \mu &= 100.\end{aligned}$$

Those parameter data correspond to water of 20° C except for σ . The value of σ was taken to be smaller than the proper value by a factor of 10^4 . This can be interpreted as taking the surface tension very small. A larger surface tension tends to stabilize the fluid flow. Hence using a larger surface tension a bifurcation would have appeared later, leaving other characteristics of the fluid flow unchanged qualitatively.

We have used as bifurcation parameter the Reynolds number λ , which we varied between $\lambda = 10^{-2}$ and $\lambda = 1.5 \cdot 10^4$. The Reynolds number is given by

$$\lambda = \frac{u_R d_R}{\nu}, \quad (5.40)$$

where d_R and u_R are the reference values for the length and velocity, respectively. We have chosen as reference length

$$d_R = 1 \text{ cm}.$$

Then the Reynolds number

$$\lambda = 10^3$$

is equivalent to a reference velocity

$$u_R = 0.1 \frac{\text{m}}{\text{s}}.$$

The reference velocity is in our case the velocity of the basic flow at the free surface. Running the program AUTO we discovered two bifurcation points on the trivial branch. The first bifurcation point is

$$\lambda_1 = 1.945 \cdot 10^3$$

and the second is

$$\lambda_2 = 1.315 \cdot 10^4.$$

Starting from λ_2 we were not able to find nontrivial solutions. From the first bifurcation point λ_1 we computed a nontrivial branch. The branch first moves to the right, turns around and goes then back to smaller λ - values. Finally at λ approximately 750 the L^2 - norm of the solution blows up. The computation ended with convergence problem. In Figures 5.1 and 5.2 the L^2 -norm of the solution is plotted versus the parameter λ . Figure 5.2 magnifies the area around the bifurcation point λ_1 , to show better how the solution branches off from the basic solution and turns then around to smaller λ -values. Figures 5.3 and 5.5 show the velocities $u^1(z)$ and $u^3(z)$ for $x = 0.5$. When we move along the nontrivial branch u^1 is monotonically decreasing for $z = 0.4$, whereas u^3 is monotonically increasing for $z = 0.2$. In this way one might spot the corresponding curves in the pictures when moving on the branch. The labels 1,2 and 3 in the bifurcation diagram correspond to the solution curves in Figures 5.4, 5.6 and 5.8. Figure 5.7 shows the free surface of the nontrivial solution. The amplitude of the wave is permanently growing when we follow the branch. At the end the maximal amplitude is already $1/3$ of the total height of the fluid.

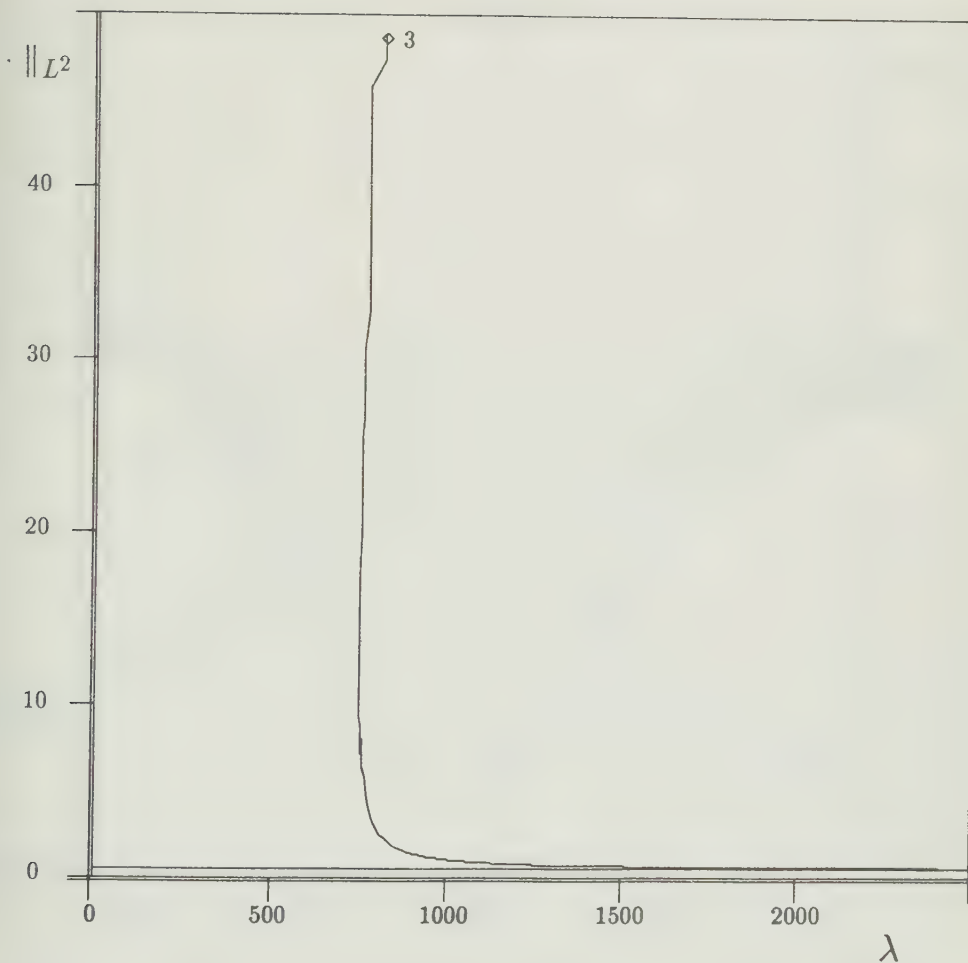


Figure 5.1: *Bifurcation diagram, L^2 - norm of the solution versus the Reynolds number λ*

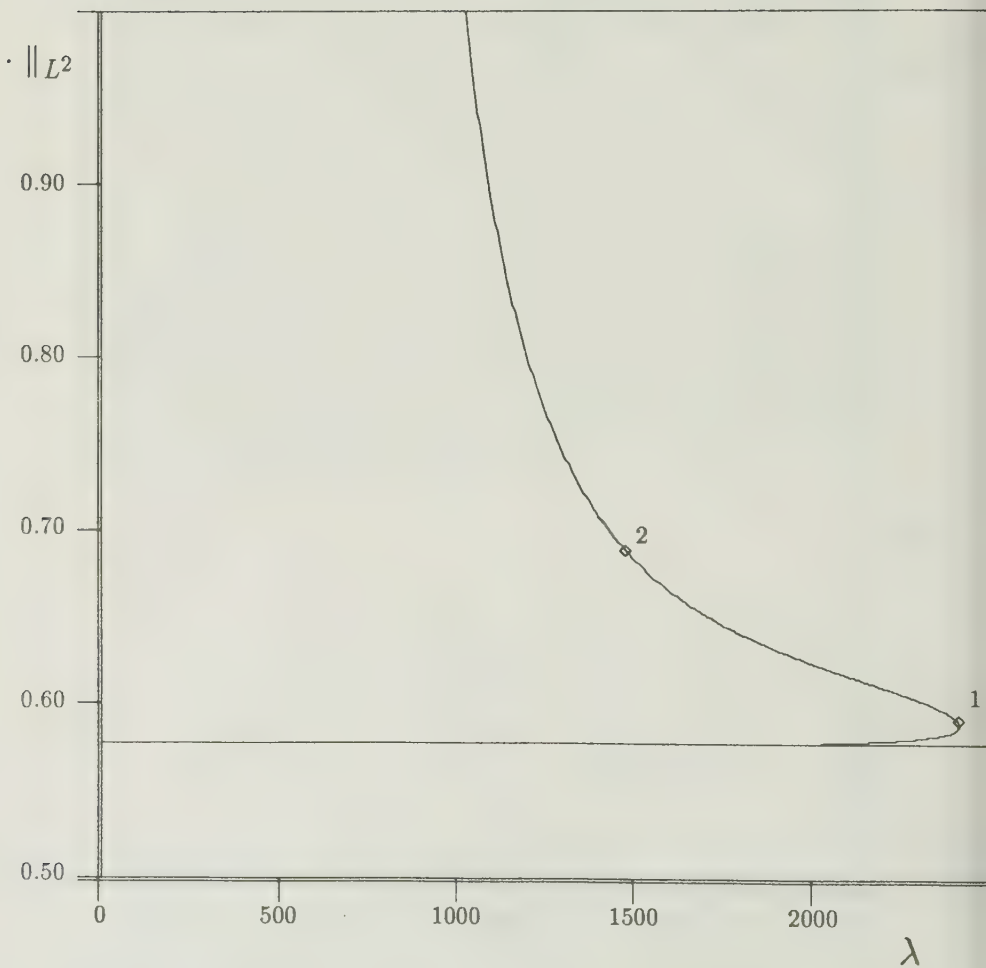


Figure 5.2: *Bifurcation diagram, L^2 - norm of the solution versus the Reynolds number λ*

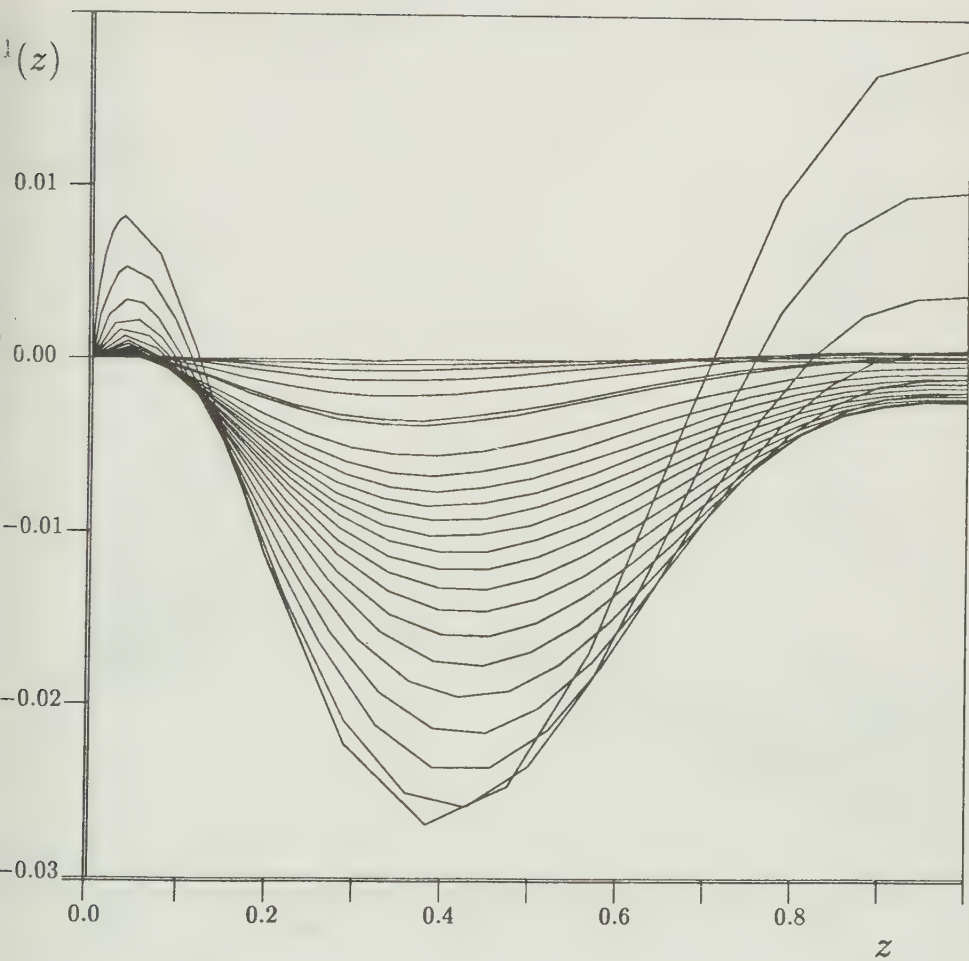


Figure 5.3: The velocity $u^1(z)$ at $x = 0.5$ for various points on the nontrivial branch

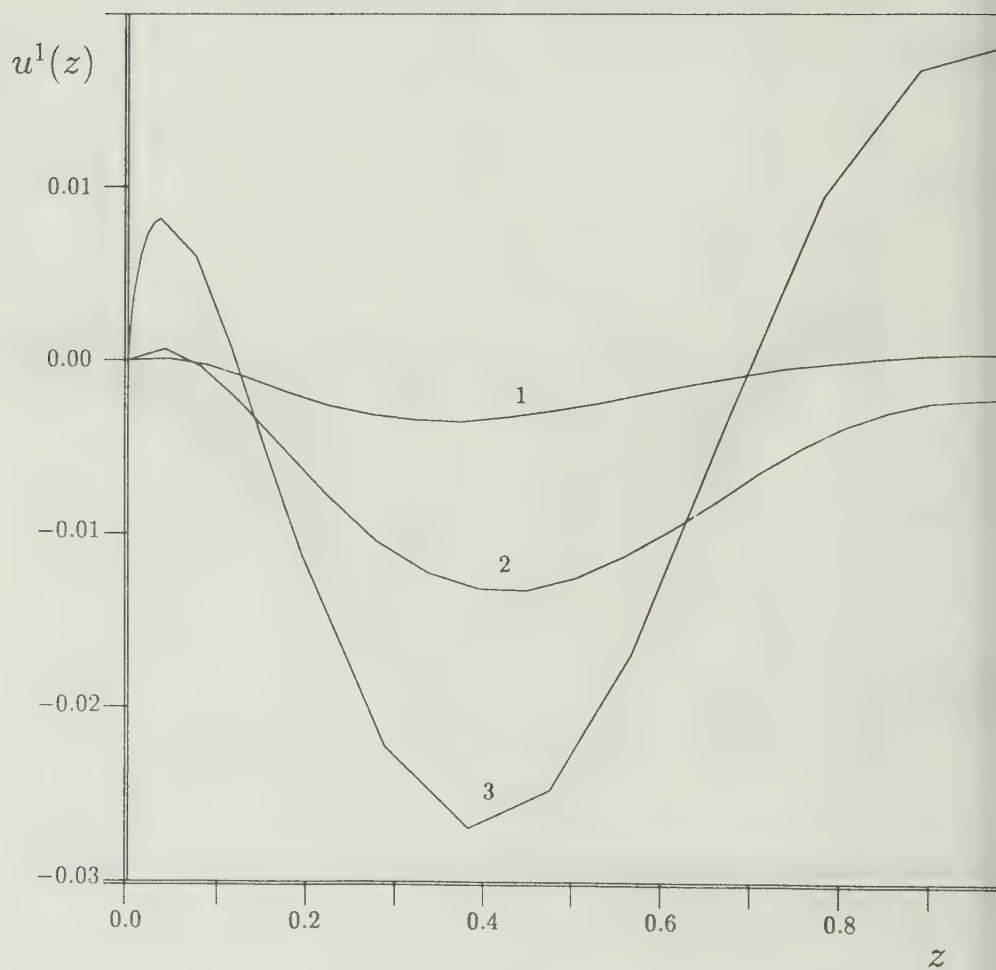


Figure 5.4: The velocities u^1 for the labeled points in the bifurcation diagram

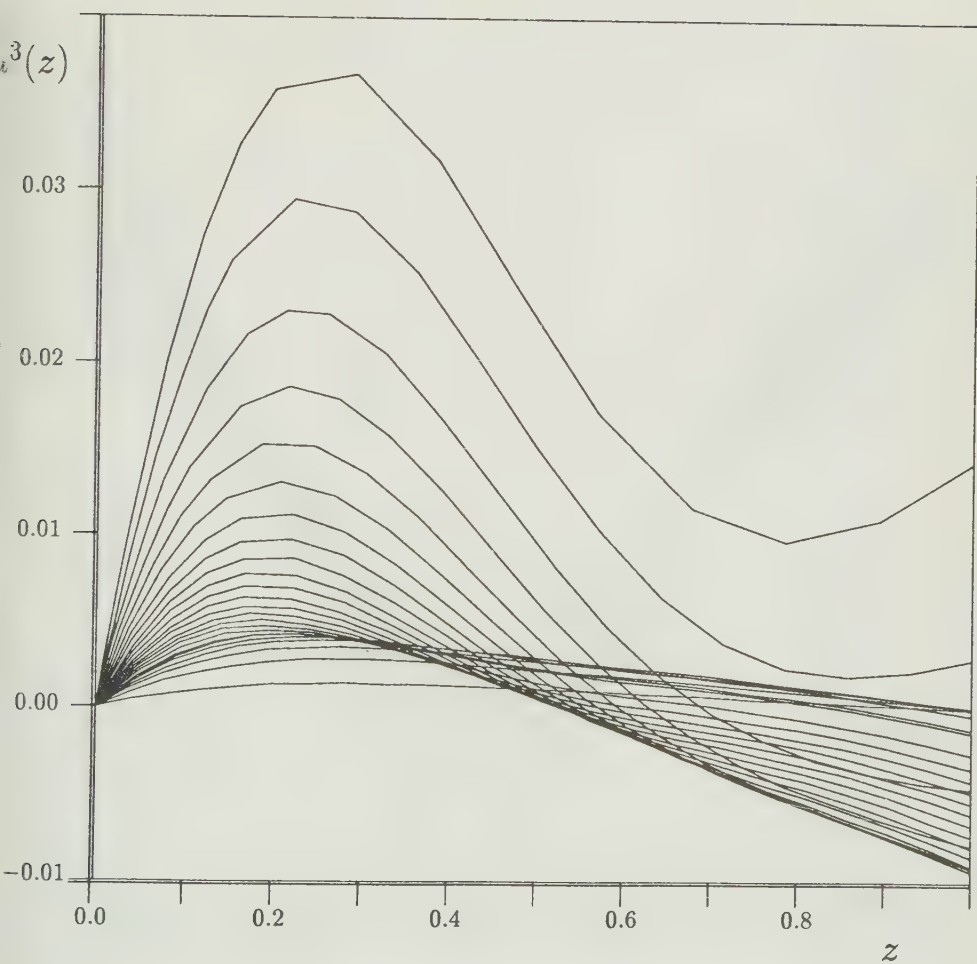


Figure 5.5: The velocity $u^3(z)$ at $x = 0.5$ for various points on the nontrivial branch

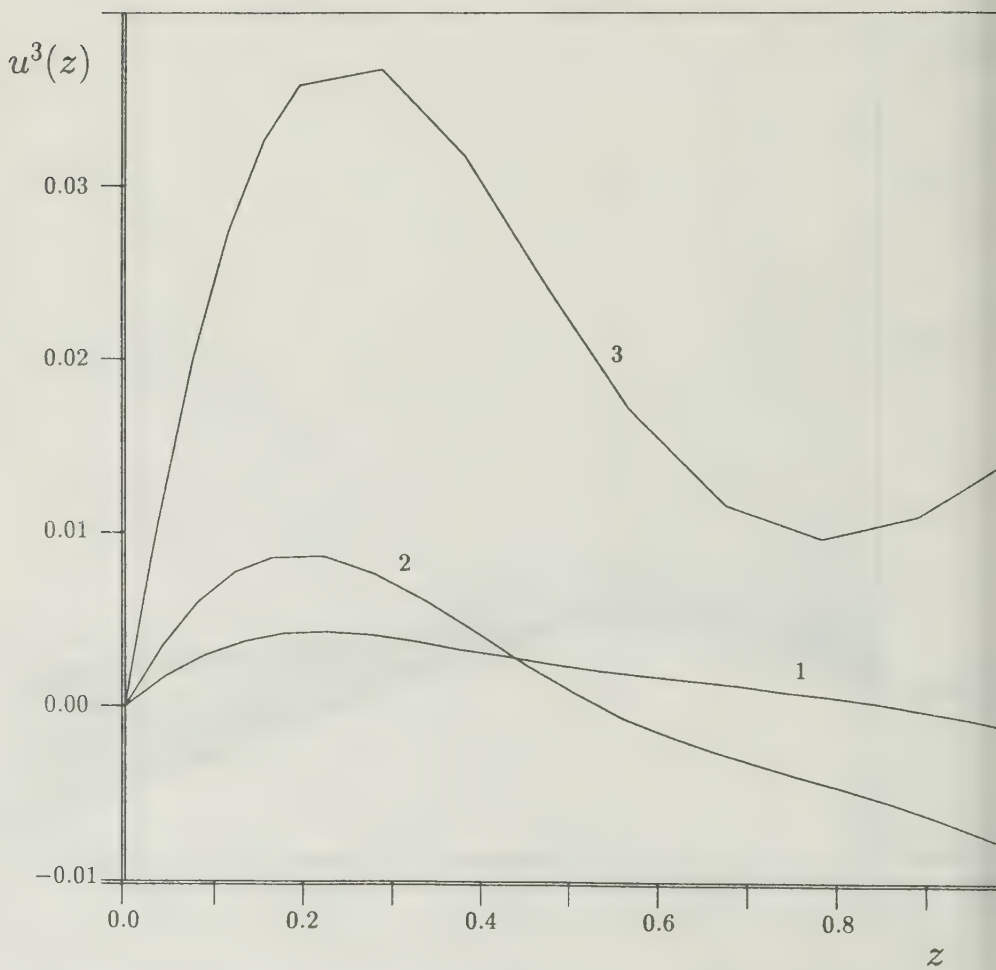


Figure 5.6: The velocities u^3 for the labeled points in the bifurcation diagram

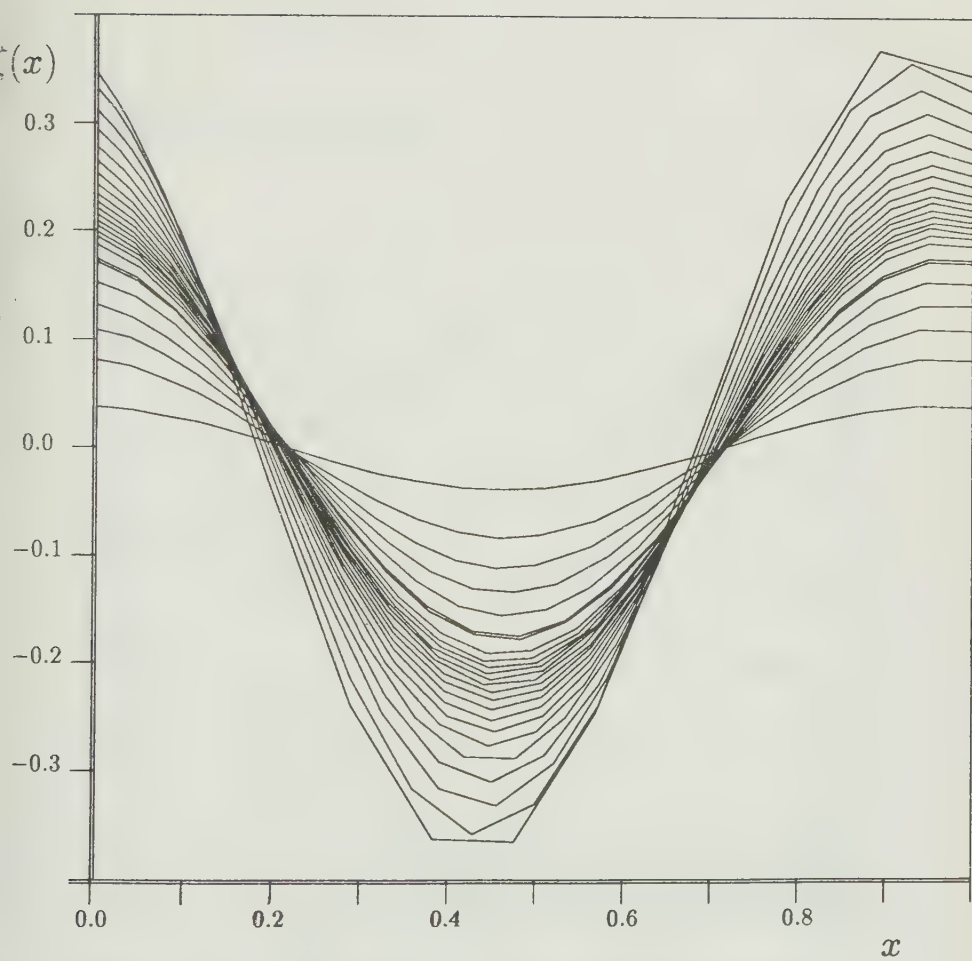


Figure 5.7: *The free surface for various points on the nontrivial branch*

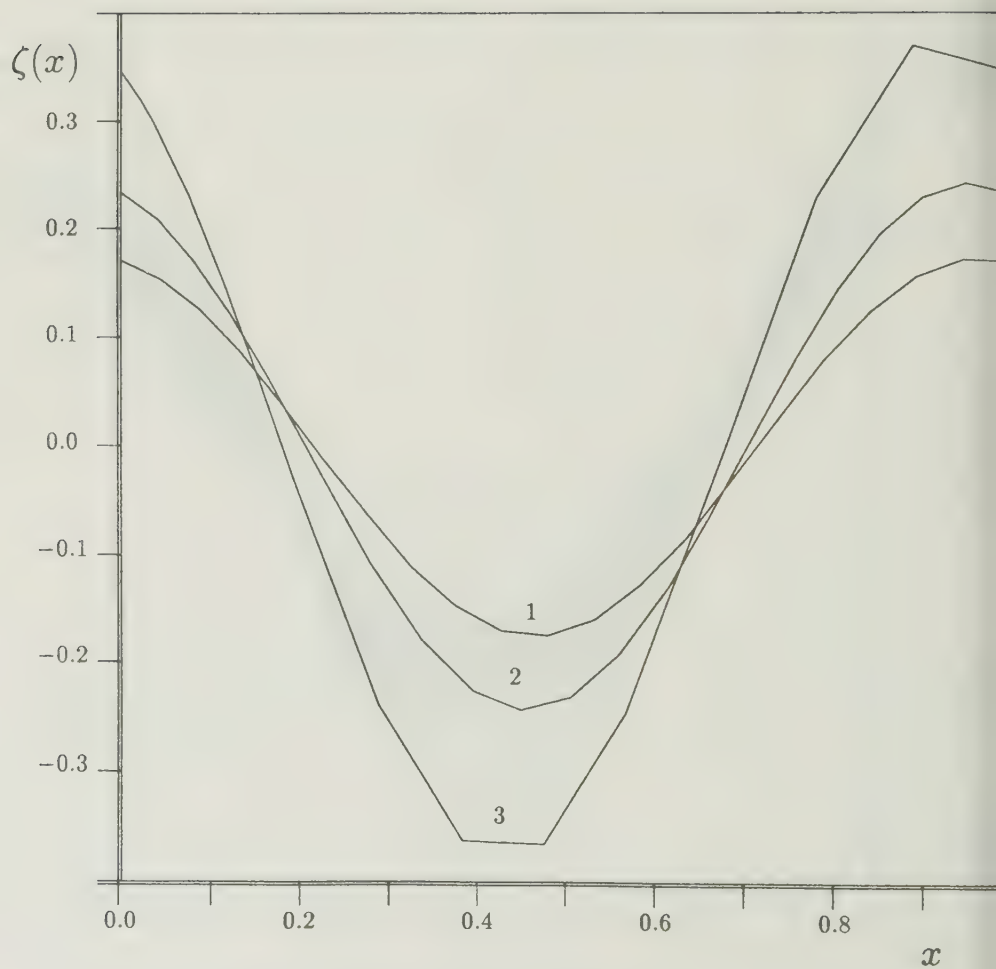


Figure 5.8: *The free surfaces for the labeled points in the bifurcation diagram*

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